Eigen-vectors and eigen-values

Let

\[ A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \]

and

\[ \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \]

Then \( \mathbf{u} = A\mathbf{v} \) is another vector:

\[ \mathbf{u} = A\mathbf{v} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 1 \times 2 \\ -1 \times 1 + 3 \times 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}. \]

**Convention:** We identify the vector \([xy]\) with the point \((x, y)\) in the \(xy\)-plane:

Consider the following scenario: If \(V\) is a collection of vectors, then \(AV := \{A\mathbf{v} : \mathbf{v} \in V\}\) is also a collection of vectors.

Geometrically, every vector (or point) in \(V\) is moved to a new position in \(AV\) according to a rule given in terms of the matrix \(A\). So, we can view a matrix multiplication to a vector as a geometrical motion.

**Question:** Can we determine all the vectors whose directions are not changed when moved by \(A\)? That is, we want to find all \(\mathbf{v}\) such that

\[ A\mathbf{v} = k\mathbf{v} \]

for some scalar \(k\) depending on \(\mathbf{v}\).

This turns out to be an important question. Note that not one but two unknown objects, \(k\) and \(\mathbf{v}\), are needed to answer the question.
**Definition:** Let $A$ be an $n \times n$ matrix. A number (scalar) $k$ is called an eigen-value of $A$ if there exists a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$A\mathbf{v} = k\mathbf{v}.$$ 

In this case, the vector $\mathbf{v}$ is called a eigen-vector of matrix $A$ corresponding to $k$.

The above definition put $k$ in a more prominent role than $\mathbf{v}$. This a side effect of the way we solve the "eigen-problem": We first determine all possible $k$ without knowing any eigen-vectors except their existence.

Note that there may exist several eigen-vectors all corresponding to the same eigen-value (see Matlab examples), and there are at most $n$ eigen-values (this is not so obvious but it can be proved with the help of the concept of determinants [still remember it?!]).

Eigen-values are sometimes also called characteristic values.

Computing the eigen-values and vectors is an important task in solving a lot of problems. In Matlab, the single command $\text{eig}(A)$ will return us both eigen-values and the corresponding eigen-vectors (isn’t it nice!).

**Help:**

```
EIG    Eigenvalues and eigenvectors.
E = EIG(X) is a vector containing the eigenvalues of a square matrix X.

[V,D] = EIG(X) produces a diagonal matrix D of eigenvalues and a
full matrix V whose columns are the corresponding eigenvectors so
that X*V = V*D.

[V,D] = EIG(X,'nobalance') performs the computation with balancing
disabled, which sometimes gives more accurate results for certain
problems with unusual scaling.

E = EIG(A,B) is a vector containing the generalized eigenvalues
of square matrices A and B.

[V,D] = EIG(A,B) produces a diagonal matrix D of generalized
eigenvalues and a full matrix V whose columns are the
corresponding eigenvectors so that A*V = B*V*D.
```
$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$[V \ D] = \text{eig}(A)$$

$$V = \begin{bmatrix} 0.7071 & -0.6000 \\ 0.7071 & 0.8000 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A*V$$

$$\text{ans} = \begin{bmatrix} 3.5355 & 1.2000 \\ 3.5355 & -1.6000 \end{bmatrix}$$

$$V*D$$

$$\text{ans} = \begin{bmatrix} 3.5355 & 1.2000 \\ 3.5355 & -1.6000 \end{bmatrix}$$

$$A*V-V*D$$

$$\text{ans} = \begin{bmatrix} 1.0e-015 * \\ 0 & 0 \\ 0 & -0.2220 \end{bmatrix}$$
Let us see one application of the eigen-vectors in representation problem. Consider the collection of all (column) vectors of dimension 2: There are infinitely many of such vectors. Together, they form a what is called vector space of dimension 2.

Note that each vector \[
\begin{bmatrix}
x \\
y
\end{bmatrix}
\] can be written as a linear combination of two fixed vectors, \( e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) and \( e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\):

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = xe_1 + ye_2.
\]

So, the two vectors \( e_1 \) and \( e_2 \) span (generate) the whole linear space of dimension 2. Such a set of two vectors is called a basis of the linear space. There are many other basis other than \( \{e_1, e_2\} \). For example,

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}, \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

is also a basis. Every basis can be used to represent all elements in the linear space as their linear combinations. Some basis are more useful in representing a specific set of vectors. How to choose the right basis in a given situation is the problem we want to solve.

For us the most interesting case is when this specific set of vectors is the collection of the column vectors in a given matrix.

Question: Develop a criterion to find the best basis that carry some significance for each vector in the basis so that we can represent all columns in a given matrix.

How to proceed?