Introduction.

Communication system design:

Transmission of information (voice, data) over path (communication channel - wired, space).

Digital increasingly attractive over analogue.

(Digital: during finite time send a waveform from a finite set of waveforms.

Analog: \(\infty\) many waveforms.

Objective: @ receiver is to determine from a noise-perturbed signal which waveform (from a finite number) was sent.

Why digital.

* Digital signals easily regenerated.
  Pulse degradation (non-ideal frequency transfer from trans., lines & circuits)
  & electrical noise or interference easily accounted for.

* Less susceptible to noise - only fully 'on' or 'off' binary analog - \(\infty\) variety. (Once distorted cannot regenerate)
- digital circuits are more reliable & cheaper

- digital techniques lend themselves readily to signal processing functions that protect against interference & jamming, or provide encryption & security.

Problems:

- v. signal processing intensive

- need to synchronize at various levels

- non-graceful degradation. SNR drops suddenly go v.good -> v.bad

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**Digital vs. Analog performance criteria.**

- evaluate performances differently.

- analog draws waveform from infinite continuum.

  - performance measured are fidelity criterion
  
  (degree output matches input)

  \[
  \text{SNR}, \text{ distortion}, \text{ MSE}
  \]

- digital comm transmits signal representing digits.

  These form finite set or alphabet known a priori to RX

  - measure now prob of incorrectly detecting a digit.
    - prob of error \( p_e \).
**SIGNALS & SPECTRA**

Deterministic: no uncertainty @ any pt. in time

Random: some degree of uncertainty

over a long period random waveform $\Rightarrow$ random process.

$\gamma$ exhibit some regularity  - probabilities  
- statistical averages.

This Probabilistic description necessary in characterizing signals in comm's.

**Periodicity.**

Signal $x(t) = x(t + T_0)$. $-\infty < t < \infty$

$t = \text{time}.$

$T_0 = \text{period of one cycle}.$

If no $T_0$ satisfies above $\Rightarrow$ non-periodic signal.

**Analogue & Discrete.**

Analogue - continuous $f^2$ of time: $x(t)$ defined all $+^.$

An electrical analog signal arises when a physical waveform (speech) $\Rightarrow$ electrical signal by transducer.

Discrete - exists @ discrete times... characterised by a sequence of no's defined for each $kT$

$k =$ integer  
$T =$ fixed time.
Energy & Power signals

Voltage $v(t)$, current $i(t)$.

Instantaneous power across $R$, $p(t)$

$$p(t) = \frac{v^2(t)}{R} \quad \text{or} \quad p(t) = \frac{i^2(t) R}{R}$$

often assume $R = 1 \Omega$

$$p(t) = x^2(t) \quad \text{whether } x(t) \text{ is a voltage or current}$$

Energy dissipated in time interval $(-T/2, T/2)$

$$E_x = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \, dt$$

& av. power

$$p_x = \frac{1}{T} E_x = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \, dt$$

Comm. system: higher energy signal more readily detected

Power determines voltages that must be applied to TX

Often consider waveform energy

$x(t)$ = "energy signal" if

$$E_x = \lim_{T \to \infty} \int_{-T/2}^{T/2} x^2(t) \, dt = \int_{-\infty}^{\infty} x^2(t) \, dt$$
In real world always transmit signals with finite energy.

To describe periodic signals (which by def exist for all time & have $\infty$ energy) & random signals with $\infty$ energy.

- Define power signals

Signal defined as "power signal" if has finite power

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \, dt.$$  

Energy & Power signals are mutually exclusive.

Energy signal: finite energy, zero average power

Power signal: finite average power, $\infty$ energy

As a general rule, periodic & random signals $\Rightarrow$ Power signals

Deterministic & non-periodic $\Rightarrow$ Energy signals.

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**Unit Impulse Function**

Unit Impulse or Dirac delta $\delta^2$.

$\infty$ large amplitude, zero pulse width & unit weight (area under pulse).

$$\int_{-\infty}^{\infty} \delta(t) = 1.$$  

$\delta(t) = 0 \quad t \neq 0 \quad \delta(t)$ unbounded @ $t = 0.$
\[ \int_{-\infty}^{\infty} x(t) \delta(t - t_0) \, dt = x(t_0). \]

Above eq. is 'sifting' or 'sampling' property of unit impulse \( \delta \).

\( \delta \) selects a sample of \( x(t) \) at \( t = t_0 \).

**Spectral density.**

Spectral density characterises distortion of signals energy in frequency domain.

Important in filtering applications.

**Energy Spectral Density.**

Need forward & inverse Fourier transforms:

\[ x(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} X(f) \, df \]

\[ X(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft} x(t) \, dt. \]

Total energy of real-valued energy signal \( x(t) \), given by

\[ E_x = \int_{-\infty}^{\infty} x^2(t) \, dt \]

Parseval's Th:

\[ E_x = \int x^2(t) \, dt = \int |X(f)|^2 \, df. \]
\[ \Psi_x(f) = |X(f)|^2 = \text{"energy spectral density"} \]

\[ \therefore E_x = \int_{-\infty}^{\infty} \Psi_x(f) \, df \]

- Energy spectral density is the signal energy per unit bandwidth.

- There are equal contributions from both +ve and -ve frequencies since for real \( x(t) \), \( |X(f)| \) is an even function.

\[ \therefore \text{energy spectral density is symmetric about origin} \]

& total energy

\[ E_x = 2 \int_{0}^{\infty} \Psi_x(f) \, df \]

**Power Spectral Density**

\[ P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t) \, dt \]

Parseval's theorem for a real valued periodic signal is

\[ P_x = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x^2(t) \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \quad (\text{since } x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nf_0 t}) \]

The PSD \( P_x \), \( G_x(f) \), of \( x(t) \) is real, even, & a non-negative \( P_x \) of frequency

\[ G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f-nf_0) \]

\( \therefore \) PSD of sinusoid is discrete. Generally \( f_0 \) sequence of
\[ P_x = \int_{-\infty}^{\infty} G_x(f) \, df = 2 \int_{0}^{\infty} G_x(f) \, df \]

Above is for periodic only, i.e. Fourier series. For non-periodic may express PSD in limiting sense.

Form truncated version of \( x(t) \), i.e. \( x_T(t) \). Then \( x_T(t) \) has finite energy, & FT \( x_T(f) \)

Then \[ G_x(f) = \lim_{T \to \infty} \frac{1}{T} \left| x_T(f) \right|^2. \]

**Autocorrelation of energy signal**

Matching of a signal with a delayed version of itself,

\[ R_x(\tau) = \int_{-\infty}^{\infty} x(t) x(t + \tau) \, dt \quad -\infty < \tau < \infty. \]

\( R_x(\tau) \) measure of how closely \( x(t) \) matches copy of itself shifted by \( \tau \).

\( R_x(\tau) \) for of time diff. only.

**Properties**

1. \( R_x(\tau) = R_x(-\tau) \). Symmetric about origin
2. \[ |R_x(\tau)| \leq R_x(0) \quad \text{max at origin} \]
3. \( R_x(\tau) \leftrightarrow G_x(f) \). FT pair.
4. \[ R_x(0) = \int_{-\infty}^{\infty} x^2(t) \, dt \]. Value at origin = energy of signal.

F follows 3.
Autocorrelation of periodic signal

\[ R_x(t) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x(t+\tau) \, dt \quad -\infty < \tau < \infty \]

If \( x(t) \) periodic with period \( T_0 \)

\[ R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) x(t+\tau) \, dt. \]

Properties 1 & 2 same

3. \( R_x(\tau) \leftrightarrow G_x(\lambda) \)
4. \( R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t) \, dt. \)

Random Signals

Need efficient descriptions of random signals.

Random Variables.

RV \( X(A) = \) function relating between event \( A \) & a real no. \( X \).

Usually drop \( A \) - assume dependence implicit.

Distribution Function:

\[ F_X(x) = P(X \leq x) \]

\[ P(X \leq x) = \text{prob}\{X \leq x\} \quad X \text{ is less than or equal to real no. } x. \]
Properties.

1. \(0 \leq F_X(x) \leq 1\)
2. \(F_X(x_1) \leq F_X(x_2)\) if \(x_1 \leq x_2\).
3. \(F_X(-\infty) = 0\)
4. \(F_X(+\infty) = 1\).

Prob density \(f_x\)

\[ p_X(x) = \frac{dF_X(x)}{dx} \]

Named "density \(f^x\)" since prob of event \(x_1 \leq X \leq x_2\) is:

\[
P(x_1 \leq X \leq x_2) = P(X \leq x_2) - P(X \leq x_1) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} p_X(x) \, dx
\]

Note: \(P(x \leq X \leq x+\Delta x) \approx p_X(x) \, \Delta x\)

\[
\therefore P(X = x) = p_X(x) \, dx.
\]

Properties:

\(p_X(x) \geq 0\)

\[
\int_{-\infty}^{\infty} p_X(x) \, dx = F_X(+\infty) - F_X(-\infty) = 1
\]
Ensemble Averages

Mean or expected value of RV $X$

$$m_X = E[X] = \int_{-\infty}^{\infty} x \, p_X(x) \, dx$$

$n^{th}$ moment

$$E[X^n] = \int_{-\infty}^{\infty} x^n \, p_X(x) \, dx$$

Second mom, $n = 2$.

Central moments = diff between $X$ & $m_X$, e.g. variance.

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \int_{-\infty}^{\infty} (x - m_X)^2 \, p_X(x) \, dx$$

$$\therefore \sigma_X^2 = \text{Var}(X) = E[X^2] - m_X^2.$$ 

Random process.

$X(A,t) = f^2$ two variables; event $A$ & time.

For a specific event $A_j$, we have single time $f^2$

$$X(A_j,t) = X_j(t). \Rightarrow \text{A sample function.}$$

For a specific time $t_k$, $X(A,t_k)$ is a RV $X(t_k)$.

For $A = A_j$ & $t = t_k$, $X(A_j,t_k)$ is simply a number.

Describe random process $X(t)$, A dep. implicit.
Statistical Averages of a random process

- Value of a random process in future is unknown
- Can describe statistical behaviour with PDF
  (usually this is not available).

Partial description given by mean & autocorrelation

\[ E[X(t_1)]^2 = \int_{-\infty}^{\infty} x^2 p_{X}(x) \, dx = \mu_X(t_1) \]

\[ R_X(t_1, t_2) = E[X(t_1)X(t_2)] \]

Stationarity

\( X(t) \), Random process, is strictly stationary if
statistics unaffected by shift in time.

\( X(t) \) = wide sense stationary if mean & autocorr are
unaffected by shift in time.

Strict implies wide sense NOT vice versa.

Autocorr of WSS random process

Variance provides measure of randomness for RV's.

Autocorr provides similar measure for random processes.

\[ R_X(t) = E[X(t)X(t+c)] \]
\[ -\infty < c < +\infty \]
Similar to random value, we obtain

1. $R_x(t) = R_x(-t)$
2. $|R_x(t)| \leq R_x(0)$
3. $R_x(t) \leftrightarrow G_x(f)$
4. $R_x(0) = E \mathbb{E} X^2(t) \frac{T}{2}$

**Time averaging & ergodicity**

**Ergodic Process**: Time averages equal its ensemble averages.

Statistical properties can be found by time averaging over a single sample $x(t)$ of the process.

**Ergodic in mean**:

$$m_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, dt.$$  

**Ergodic in autocorrelation**

$$R_x(t) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x(t+t) \, dt.$$  

In electrical systems:

1. $m_x = E \mathbb{E} X(t) \frac{T}{2} = \text{d.c. of signal}$
2. $m_x^2 = \text{normalized power in d.c component}$
3. $E \mathbb{E} X^2(t) \frac{T}{2} = \text{total av. normalized power}$
PSD & autocorrelation of a random process.

As with random signal:

\[ \text{PSD} = G_x(f). \]

1. \( G_x(f) \geq 0 \) & always real
2. \( G_x(f) = G_x(-f) \) for \( x(t) \) real valued
3. \( G_x(f) \leftrightarrow R_x(r) \)
4. \( P_x = \int_{-\infty}^{\infty} G_x(f) \, df. \)

Example.

Noise inComm. systems.

Noise = any unwanted (electrical) contribution.

Noise obscures/masks signal - manmade or natural.

Can eliminate substantially - filtering
- shielding
- choice of mod\(z\) scheme.

(e.g. radio astronomy in deserts - far from man made sources).

- Thermal or Johnson noise cannot be eliminated.
  - thermal motion of e\(^{-}\)'s in dissipative elements
  - same e\(^{-}\)'s that give current create noise.
We describe thermal noise as a zero-mean Gaussian random process.

\[ n(t) = \text{random noise} \]
\[ \sigma^2 = \text{variance of noise.} \]

Gaussian pdf

\[ p(n) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{n}{\sigma} \right)^2 \right] \]

Often the signal is the sum of Gaussian noise & a d.c. component

\[ s = a + n \leq \text{noise.} \]
\[ \uparrow \text{d.c.} \]

\[ \Rightarrow p(s) = \frac{1}{\sigma \sqrt{2\pi}} \left[ -\frac{1}{2} \left( \frac{s-a}{\sigma} \right)^2 \right] \]

**White Noise**

Primary spectral characteristic of thermal noise is that its PSD is same for all freq's.

ie, thermal source emanates same noise power per unit bandwidth from d.c up to 10^2 Hz.

\[ \Rightarrow \text{Simple model assumes PSD, } G_n(f) \text{ is flat} \]

\[ G_n(f) = \frac{N_0}{2} \]
2 introduced as \( G_n(t) \) is two sided.

Uniform PSD \( \Rightarrow \) "White Noise".

Auto-correlation of white noise given by IFT of above

\[
R_n(t) = \mathcal{F}^{-1} \{ G_n(f) \} = \frac{N_0}{2} \delta(t).
\]

Average power of white noise is \( \infty \), since BW is \( \infty \)

\[
\Rightarrow P_n = \int_{-\infty}^{\infty} G_n(f) \, df = \int_{-\infty}^{\infty} \frac{N_0}{2} \, df = \infty.
\]

* No noise truly white \( \Rightarrow \) good abstraction.

So long as BW noise \( \gg \) BW system, consider noise white.

* Note noise is \( S \) correlated \( \Rightarrow \) effects transmitted info. independently.

* Additive process \( \Rightarrow \) simply superimposed on signal.

**Signal Transmission Through Linear System.**

Now consider characteristic of systems.
Can describe in time domain or frequency

\[
L[c x_1(t) + b x_2(t)] = a d \{ x_1(t) \} + b d \{ x_2(t) \}.
\]

Time invariant \( \Rightarrow \) shift time no.
**Impulse response** \( h(t) = \text{output when input} = \delta(t) \)

\[ y(t) = h(t) \text{ when } x(t) = \delta(t) \]

⇒ how does system respond to a "whack"?

From linear system theory, we find the response to an arbitrary input \( x(t) \) is

\[ y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau. \]

System causal when there can be no output before input at, say, \( t = 0 \)

\[ y(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) \, d\tau. \]

\[ \text{convolution integral.} \]

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**Frequency Transfer \( f^2 \)**

In frequency domain

\[ Y(f) = X(f) \cdot H(f) \quad \text{⇒} \quad H(f) = \frac{Y(f)}{X(f)} \]

\[ H(f) = \mathcal{F}\{h(t)\} = \text{frequency transfer } f^2. \]

In general \( H(f) \) is complex & can be written.

\[ H(f) = |H(f)| \cdot e^{j\Phi(f)} \]

\[ |H(f)| = \text{magnitude response} \]

\[ \Phi(f) = \text{phase response} = \tan^{-1} \left( \frac{\text{Im} \{H(f)\}}{\text{Re} \{H(f)\}} \right) \]
Input & Output Power Spectral densities related as:

\[ G_y(f) = G_x(f) |H(f)|^2 \]

**Distortionless Transmission**

Ideal transmission:
- some time delay
- different amp
- no change in shape

\[ y(t) = k x(t - t_0). \]

\[ y(f) = k X(f) e^{-j2\pi f t_0} \]

\[ H(f) = k e^{-j2\pi f t_0} \]

Ideal distortionless transmission requires system response with constant magnitude & linear phase shift with frequency.

One cannot build above network as would require infinite BW per all f's.

As an approx., we choose truncated network that passes, without distortion, all f's between \( f_1 \) & \( f_u \).

\( f_1 = \text{lower cutoff} \)  
\( f_u = \text{upper cutoff} \)
3 Types

\[ H(\omega) \]

- BP: \(-f_b\) to \(f_b\)
- LP: \(-f_L\) to \(f_L\)
- HP: \(-f_L\) to \(f_L\)

Realizable Filters

Low Pass

Simolest case is a resistor \(R\) & capacitance \(C\).

\[ x(t) = RC \frac{dy(t)}{dt} + y(t) \]

\[ \implies FT \text{ above} \]

\[ \mathcal{F}\{\frac{dy(t)}{dt}\} \] \( = \) \( j2\pi f RC \)

\[ \implies \mathcal{F}\{x(t)\} = \frac{1}{j2\pi f RC + 1} \]

\[ H(f) = \frac{1}{1 + j2\pi f RC} = \frac{1}{\sqrt{1 + (2\pi f RC)^2}} \cdot e^{-j\phi(f)} \]

\[ \phi(f) = \tan^{-1} 2\pi f RC \]
Several useful approximations to ideal low-pass filters. One of these is the Butterworth filter:

\[ |H_n(f)| = \frac{1}{\sqrt{1 + (f/f_u)^{2n}}} \quad n \geq 1. \]

\[ f_u = -3\text{dB} \text{ cutoff}. \]