Probability

**Experiment** is a process that results in an observation that cannot be determined with certainty in advance of the experiment. Each observation is called an outcome or a sample point which may be qualitative or quantitative.

Examples of experiments include (i) Toss a coin (H or T) once, (ii) Toss a coin twice, (iii) through a die (1, 2, 3, 4, 5, 6) once, (iv) Through a die twice, (v) Select \( n \) individuals at random from a group of \( N \) individuals, (vi) Draw cards from a deck of 52 cards, (vii) Observe a day for rain, no rain (viii) Collect a persons’s opinion (for, against) towards a government policy, (ix) Examine a person for the presence or absence of a given disease, etc.

**Sample Space (\( S \))** of an experiment is the set of all possible outcomes or sample points of the experiment. (Recall and compare with the definition of population)

**Event (\( A \))** is a subset of the sample space of an experiment. (Recall and compare with the definition of sample)

**Probability of an event \( A \)**

\[
P(A) = \text{sum of the probabilities of all sample points that are in } A.
\]

Probability Rule:
1. Probability of each sample point lies between 0 and 1.
2. Sum of the probabilities of all sample points of a sample space is equal to 1.

Remark:
- **Equally Likely Points** means each point in the sample space has equal probability.
- **Random selection** imply equally likely outcomes.
- **Fair coin** means equal probability of \( 1/2 \) for Heads and \( 1/2 \) for Tails.
- **Unbiased or Fair Die** means equal probability of \( 1/6 \) for each of the six possible points.
- **Proportions** from percentages (often found in journals, newspapers, etc.) are used as probabilities of sample points.]
Example. Two fair coins are tossed, and their up faces (H or T) are recorded. Write down the sample space $S$ and the probabilities of all sample points. Consider the following two events:

- $A$: observe exactly one head
- $B$: observe at least one head

Write down the sample points that are in $A$ and $B$. Find the $P(A)$, and $P(B)$. 
Union, Intersection, and Complement of Events

Let $S$ be the Sample Space, $A$ and $B$ be two subsets of $S$.

- The union of two events $A$ and $B$, written as $A \cup B$, consists of all points of $S$ that belong to either $A$ or $B$ or both.
- The intersection of $A$ and $B$, written as $A \cap B$, consists of all points of $S$ that belong to both $A$ and $B$.
- Two events $A$ and $B$ are said to be disjoint or mutually exclusive if $A \cap B$ is an empty set.
- The complement of $A$, written as $A^c$, consists of all points of $S$ that are not in $A$.

Laws of Probability

Addition Law: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Complementary Law: $P(A) + P(A^c) = 1$

Example.

Recall the previous coin tossing experiment. Two fair coins are tossed, and their up faces (H or T) are recorded.

$S = \{HH, HT, TH, TT\}$

$A = \{HT, TH\}$

$B = \{HH, HT, TH\}$

Find $A \cup B$, $A \cap B$, $P(A \cup B)$, $P(A \cap B)$, $P(A^c)$.

Example.

The weather at a Caribbean island is perfect for a tourist destination. The chance that it rains on any particular day is only 20% and each day’s weather is independent of every other day’s weather. Suppose you plan a vacation at this island for three days. What is the probability that you get rain on at least one day? one day? no day? at most two days?
Example

The Statistical Abstract of the United States reports that 25% of the country’s households are composed of one person. If 3 randomly selected homes are to participate in a Nielsen survey to determine television ratings, find the probability that (i) fewer than two of these homes are one-person households, (ii) at least one home is one-person households.

Example

The following information is obtained from a group of 1000 students.

Distribution of students according to class rank and employment status

<table>
<thead>
<tr>
<th></th>
<th>Full-time</th>
<th>Part-time</th>
<th>Unemployed</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fr</td>
<td>100</td>
<td>200</td>
<td>10</td>
<td>310</td>
</tr>
<tr>
<td>So</td>
<td>20</td>
<td>80</td>
<td>50</td>
<td>150</td>
</tr>
<tr>
<td>Jr</td>
<td>50</td>
<td>75</td>
<td>25</td>
<td>150</td>
</tr>
<tr>
<td>Sr</td>
<td>70</td>
<td>40</td>
<td>280</td>
<td>390</td>
</tr>
<tr>
<td>Total</td>
<td>240</td>
<td>395</td>
<td>365</td>
<td>1000</td>
</tr>
</tbody>
</table>

If a student is selected at random from these 1000 students, what is the probability that the student is (i) senior? (ii) unemployed? (iii) senior and unemployed?

Example

The following table summarizes the race and positions of 368 National Basketball Association (NBA) players in 1993 (Sociology of Sport Journal, Vol 14, 1997). Suppose an NBA player is randomly selected from that year’s player pool.

<table>
<thead>
<tr>
<th></th>
<th>Guard</th>
<th>Forward</th>
<th>Center</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
<td>26</td>
<td>30</td>
<td>28</td>
<td>84</td>
</tr>
<tr>
<td>black</td>
<td>128</td>
<td>122</td>
<td>34</td>
<td>284</td>
</tr>
<tr>
<td>Totals</td>
<td>154</td>
<td>152</td>
<td>62</td>
<td>368</td>
</tr>
</tbody>
</table>

(a) What is the probability that the player plays guard but is not African-American?

(b) What is the probability that the player is white or a center?

Example

Let A and B be two subsets of the sample space of an experiment. If P(A) = 0.4, P(B) = 0.5, and P(A ∩ B) = 0.1, find (using Venn diagram or otherwise)

(a) P(A ∩ B)

(b) P(A∪B)

Example

A committee of two students is to be formed at random from a group of five students (three male students M1, M2, M3 and two female students F1, F2). What is the probability of selecting (i) both male students? (ii) exactly one male student? (iii) at least one female student?

Example
In a factory, machine A produces 25% of all items, machine B 40%, and machine C 35%. Machine A produces 5% defective items, 4% of the items produced by machine B are defective, 3% of the items produced by machine C are defective. What is the probability that a randomly selected item is defective? (Answer: 0.039)

Example
One important finding of a recent study was that 7.8% of children with nonsmoking parents had episodes of pneumonia and/or bronchitis in the first year of life, whereas, 11.4% of children with one smoking parent and 17.6% of children with two smoking parents had such an episode. Suppose that in the general population both parents are smokers in 40% of the households, one parent smokers in 25% of households, and neither parent smokers in 35% of households. What percentage of children in the general population will have pneumonia and/or bronchitis in the first year of life? What is the probability that a randomly selected newborn will have pneumonia and/or bronchitis in the first year of life? (Answer: 0.1262)

• **Conditional Probability** of an event A given that (or knowing that) event B already occurred is defined by

\[ P(A/B) = \frac{P(A \cap B)}{P(B)} \]

• **Conditional Probability** of an event B given that (or knowing that) event A already occurred is defined by

\[ P(B/A) = \frac{P(A \cap B)}{P(A)} \]

• **Multiplicative Rule of Probability**

\[ P(A \cap B) = P(B)P(A/B) = P(A)P(B/A) \]

• **Independent Events.** Two events A and B are independent if and only if \( P(A \cap B) = P(A)P(B) \).

Example.
Two pass-fail exams (one in physics and one in mathematics) are given to each of 200 students. The results are summarized in the following table.

<table>
<thead>
<tr>
<th>Pass in Physics</th>
<th>Fail in Physics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pass in Math</td>
<td>45</td>
</tr>
<tr>
<td>Fail in Math</td>
<td>35</td>
</tr>
</tbody>
</table>

A student is selected at random from 200 students.

a. What is the probability that the student passed math?

b. What is the probability that the student passed math given that the student failed physics?

c. It is known that the student failed in physics. What is the probability that the student passed math?

d. The selected student is found to have passed math. What is the probability that the student failed physics?

e. What is the probability that the student passed both math and physics?

f. Are two events "Fail Math" and "Pass Physics" independent?

Example.
The following information is obtained from a group of 1000 students.

| Distribution of students according to class rank and employment status |
|-----------------------------|-------------------|-----------------|-----------------|
|                             | Full-time | Part-time | Unemployed | Total |
| Fr                          | 100       | 200       | 10          | 310   |
| So                          | 20        | 80        | 50          | 150   |
| Jr                          | 50        | 75        | 25          | 150   |
| Sr                          | 70        | 40        | 280         | 390   |
| Total                       | 240       | 395       | 365         | 1000  |

(a) If a student is selected at random from these 1000 students, what is the probability that the student is senior as well as unemployed?

(b) If a student selected at random from these 1000 students is found to be senior, what is the probability that the student is unemployed?
The following table summarizes the race and positions of 368 National Basketball Association (NBA) players in 1993 (Sociology of Sport Journal, Vol 14, 1997). Suppose an NBA player is randomly selected from that year’s player pool.

<table>
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<tr>
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<th>Forward</th>
<th>Center</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
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<td>White</td>
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<td>black</td>
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<td>122</td>
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</tr>
<tr>
<td>Totals</td>
<td>154</td>
<td>152</td>
<td>62</td>
</tr>
</tbody>
</table>

(a) What is the probability that the player plays guard but is not African-American?

(b) What is the probability that the player is white or a center?

(c) What is the probability that the player is a center given that the player is African-American?

(d) Are the events {white player} and {Center} independent?

Counting Rules

1. Product Rule:
   If an experiment has \( n_1 \) possible outcomes, and a second experiment has \( n_2 \) possible outcomes, then the two experiments result in \( n_1n_2 \) outcomes. This can be generalized to any number of experiments.
   Stated differently:
   If a job can be done in \( n_1 \) ways, and a second job can be done in \( n_2 \) ways, then the two jobs can be done in \( n_1n_2 \) ways. This can be generalized to any number of jobs.

2. Combinations Rule:
   The number of ways \( n \) units (subjects, items, etc.) can be selected from \( N \) distinctly different units without regard to their order of selection is equal to
   \[
   \binom{N}{n} = \frac{N!}{n!(N-n)!}
   \]
   where \( n! = 1.2.3.....n \) and is called \( n \) factorial.

3. Permutations Rule:
   The number of ways \( n \) units (subjects, items, etc.) can be selected from \( N \) distinctly different units and arrange them within \( n \) positions is equal to
   \[
   \pi \! \! \! n \! \! \! N = \frac{N!}{(N-n)!}
   \]

Example.
A product can be shipped by four airlines, and each airline can ship via three different routes. How many distinct ways exist to ship the product?

Example
A committee of two students is to be formed at random from a group of five students (three male students M1, M2, M3 and two are female students F1, F2). What is the probability of selecting (i) both male students? (ii) exactly one male student? (iii) at least one female student?

Example
The republican governor of a state with no income tax is appointing a committee of five members to consider changes in the income tax law. There are 15 state representatives available for appointment to the committee, seven Democrats and 8 Republicans including governor’s friend Mr. XXX. Assume that the governor selects the committee of five members randomly from the 15 representatives.

(a) In how many different ways can the committee members be selected?
(b) What is the probability that no Democrat is appointed to the committee?

(c) What is the probability that only two Democrats are appointed to the committee?

(d) What is the probability that Mr. XXX is appointed to the committee?

(e) What is the probability that three Republicans including Mr. XXX is appointed to the committee?

Review Problems

Practice problems:
1. A lot consisting 50 bulbs is inspected by taking at random 10 bulbs and testing them. If the number of defective bulbs is at most 1, the lot is accepted; otherwise it is rejected. If there are 10 defective bulbs in the lot, what is the probability of accepting the lot?

2. In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers at random. A player has one ticket. What is the probability that the player has (a) all 8 of the number selected? (b) seven of the 8 numbers selected? (c) at least 6 of the 8 numbers selected?

3. From a group of 3 freshmen, 4 sophomores, 4 juniors, and 3 seniors a committee of size four is randomly selected. Find the probability that the committee will consists of one from each class?

4. Urn I contains 2 white and 4 red balls, whereas urn II contains 1 white and 1 red ball. A ball is randomly chosen from urn I and put into urn II, and a ball is then randomly selected from urn II. What is the probability that the ball selected from urn II is white? What is the probability that the transferred ball was white, given that a white ball is selected from urn II?

5. You ask your neighbor to water a sickly plant while you are on vacation. Without water, it will die with probability 0.8, with water it will die with probability 0.15. You are 90% certain that your neighbor will remember to water the plant. What is the probability that the plant will be alive when you return? If it is dead, what is the probability your neighbor forgot to water it?

6. A parallel system functions whenever at least one of its components works. Consider a system of 3 components and suppose that each component independently works with probability 0.40. Find the probability that the component 1 works given that the system is functioning.

7) Answer “TRUE” if the statement is always true. Otherwise answer “False”

—— If $A$ and $B$ are any two events defined on a sample space $S$ of an experiment, then $p(A \cap B) = p(A)p(B)$

—— If $A$ and $B$ are two disjoint events defined on a sample space $S$ of an experiment, then $p(A \cup B) = p(A) + p(B)$.

8) In a factory, machine $A$ produces 25% of all items, machine $B$ 40%, and machine $C$ 35%. Machine $A$ produces 5% defective items, 4% of the items produced by machine $B$ are defective, 3% of the items produced by machine $C$ are defective. What is the probability that a randomly selected item is defective? A randomly selected item is found to be defective, what is the probability that it was produced by machine $A$?

9) A computer retail store has nine personal computers (PCs) in stock. A university buyer plans to purchase three of them. Unknown to either buyer or seller, two of the PCs in stock have defective disk drives. Three computers are selected at random from the nine available. What is the probability that at least one of the computers selected will have a defective disk drive?

10) Psychologists tend to believe that there is a relationship between aggressiveness and order of birth. To test this belief, a psychologist chose 500 elementary school students at random and administered each a test designed to measure the student’s aggressiveness. Each student was classified according to one of four categories. The percentages of students falling in the four categories are shown here.

<table>
<thead>
<tr>
<th></th>
<th>Firstborn</th>
<th>Not Firstborn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggressive</td>
<td>75</td>
<td>75</td>
</tr>
<tr>
<td>Not Aggressive</td>
<td>125</td>
<td>225</td>
</tr>
</tbody>
</table>

(a.) If one student is chosen at random from the 500, what is the probability that the student is firstborn?

(b.) What is the probability that the student is aggressive, given that the student was firstborn?

(c.) If $A : \{ \text{Student chosen is aggressive}\}$, and $B : \{ \text{Student chosen is firstborn}\}$, are $A$ and $B$ independent? Show your work to support your claim.
10E) Suppose the data in exercise 10 are as follows:

<table>
<thead>
<tr>
<th>Firstborn</th>
<th>Not Firstborn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggressive</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>90</td>
</tr>
<tr>
<td>Not Aggressive</td>
<td>140</td>
</tr>
<tr>
<td></td>
<td>210</td>
</tr>
</tbody>
</table>

If \( A \) : \{ Student chosen is aggressive \}, and \( B \) : \{ Student chosen is firstborn \}, are \( A \) and \( B \) independent? Show your work to support your claim.

11) A manufacturer of 35-mm cameras knows that a shipment of 34 cameras sent to a large discount store contains nine defective cameras. The manufacturer also knows that the store will choose two of the cameras at random, test them, and accept the shipment if neither is defective. Find the probability that both the cameras selected are defective.

2. In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers at random. A player has one ticket. What is the probability that the player has (a) all 8 of the number selected? (b) seven of the 8 numbers selected? (c) at least 6 of the 8 numbers selected?

12) Investing is a game of chance. Suppose there is a 35% chance that a risky stock investment will end up in a total loss of your investment. Because the rewards are so high, you decide to invest in three independent risky stocks. What is the probability that all three stocks end up in total losses?

13) A human gene carries a certain disease from the mother to the child with a probability rate of 60%. That is, there is a 60% chance that the child becomes infected with the disease. Suppose a female carrier of the gene has four children. Assume that the infections of the four children are independent of one another. Find the probability that none of the children get the disease from their mother.

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

14) In how many ways can a committee of three men and four women be formed from a group of 10 men and 10 women?
   A) 70
   B) 50400
   C) 3,628,800
   D) 25200

15) Which of the following probabilities for the sample points A, B, and C could be true if A, B, and C are the only sample points in an experiment?
   A) \( P(A) = 1/8, P(B) = 1/7, P(C) = 1/6 \)
   B) \( P(A) = 1/4, P(B) = 1/4, P(C) = 1/4 \)
   C) \( P(A) = -1/4, P(B) = 1/2, P(C) = 3/4 \)
   D) \( P(A) = 0, P(B) = 1/15, P(C) = 14/15 \)

16) Fill in the blank. A(n) \( \text{———} \) is the process that leads to a single outcome that cannot be predicted with certainty.
   A) sample space
   B) sample point
   C) event
   D) experiment

Answers:
1. \( \frac{10C_8 \cdot 40C_10 + 10C_4 \cdot 40C_8}{40C_{10}} \)
2. \( \frac{5C_8 \cdot 32C_4}{40C_8} \)
3. \( \frac{8C_4 \cdot 32C_2 + 8C_2 \cdot 32C_4 + 8C_8 \cdot 32C_2}{40C_8} \)
4. \( \frac{144}{14C_4} \)
5. \( (0.9)(0.85) + (0.1)(0.85) + (0.1)(0.6) + (0.05)(0.6) \)
6. \( 1 - (0.6)^3 \)
7. \( \frac{0.4}{(1-0.6)^3} \)
8. 0.039; 0.32
9. 49/84; 10. 200/500; 75/200

No; 10E. Yes; 11. \( \frac{9C_2}{34C_2} \)
12. \( (0.35)^3 \)
13. \( (0.4)^4 \)
14. \( 10C_3 \cdot 10C_4 \)
15. D
16. D

More practice problems with answers in parenthesis:
3:42 (0.3; 0.6; 0.8) , 3:47, 3:70 (0.85; 0; 0.7; not independent), 3:86 (0.406; 0.777; not independent), 3:141; 3:157, 3:179, 3:183.
Extra practice problems intended for Honors students only

1. Urn I contains 2 white and 4 red balls, urn II contains 1 white and 1 red ball, whereas Urn III contains 1 white and 2 red balls. A ball is randomly chosen from urn I and put into urn II, a ball is then randomly selected from urn II, and put into urn III. What is the probability that the ball selected from urn III is white? What is the probability that the transferred ball was white, given that a white ball is selected from urn III?

2. A parallel system functions whenever at least one of its components works. Consider a system of 4 components and suppose that each component independently works with probability 0.40. What is the probability that the system is functioning? Find the probability that the component 1 works given that the system is functioning.

3. A parallel system functions whenever at least one of its components works. Consider a system of n components and suppose that each component independently works with probability 0.40. How many components are needed to make sure that the system functions with probability 0.90?

4. A plane with four engines can fly you from A to B if at least one of its engines works properly. Suppose that engines work independently each with a failure rate of 0.01 that you will complete your journey from A to B?

5. Government officials claim that a nuclear power plant is 99% of an accident that will kill all residents within 50 miles from the plant. There are three such nuclear power plants within fifty miles from the center of city. What is the probability that all resident of the city will die due to nuclear power plant accident?

6. Urn I contains 2 white and 4 black balls. Balls are randomly selected, one at a time, until a black ball is obtained. If we assume that each selected ball is replaced before the next one is drawn, what is the probability that (i) five draws are needed? (ii) at least five draws are needed?