Chapter 10. Binomial Option Pricing I

The binomial option pricing model assumes that, over a period of time, the price of the underlying asset can move only up or down by a specified amount— that is, the asset price follows a binomial distribution. In the limit, as the time step becomes smaller, this model leads to the lognormal assumption for stock prices that underlies the Black-Scholes model.

10.1 a one-period binomial tree

Consider a European call option on the stock XYZ with a strike \( K = 40 \) and 1 year to expiration. Suppose the current spot price is \( S_0 = 41 \). The price in one year can be either \( 60 \) or \( 30 \).

Consider two portfolios:

Portfolio A: buy one call option (premium paid to be determined). In one year, the payoff will be \( 0 = \max(0, S_T - 40) \) (if \( S_T = 30 \)) or \( 20 \) (if \( S_T = 60 \)).

Portfolio B: buy \( 2/3 \) share of XYZ and borrow \( 18.462 \) at the risk free interest rate 8%. In one year, the payoff should be \( 0 = 20 - 18.462 \times e^{0.08} \) if (if \( S_T = 30 \)) or \( 20 = 40 - 20 \) (if \( S_T = 60 \)).

Therefore, an investor should have the same cash flow at time 0 for these two portfolios with identical payoffs.

The idea that positions that have the same payoff should have the same cost is called the law of one price. The call option in above example is replicated by holding \( 2/3 \) shares, which implies that one option has the risk of \( 2/3 \) shares. The value of \( 2/3 \) is the delta \( \Delta \) of the option.

The Binomial Solution: Let \( S \) denote the time-0 spot price, let \( uS \) and \( dS \) denote the TWO possible price at expiration, where \( u > d \). Let \( C_0 \) denote the price of the call option at time 0. Let \( C_u \) and \( C_d \) denote the corresponding value of the option at expiration. Let \( h \) denote the period; \( r \) denote the risk free interest rate. We want to determine the \( \Delta \) – how many shares to buy and \( B \) – how much to lend, which duplicates the option pay off.

The value of the replicating portfolio of \( \Delta \) share and \( B \) in lending (\( B \) can be negative if borrow) at time \( h \) is \( \Delta S_h + e^{rh}B \) which have two possible values:

\[
\Delta \times dS \times e^\delta + B e^{rh} = C_d \\
\Delta \times uS \times e^\delta + B e^{rh} = C_u
\]

where \( \delta \) is the continuous dividend rate. Solving these two equations lead to

\[
\Delta = e^{-\delta h} \frac{C_u - C_d}{u - d}; B = e^{-rh} \frac{uC_d - dC_u}{u - d}
\]

The call option premium should be the cost of the portfolio

\[
\Delta S + B = e^{-\delta h} \frac{C_u - C_d}{u - d} + e^{-rh} \frac{uC_d - dC_u}{u - d} = e^{-rh} \left( C_u \frac{e^{(r-\delta)h} - d}{u - d} + C_d \frac{u - e^{(r-\delta)h}}{u - d} \right).
\]

To eliminate the arbitrage, we require \( u > e^{(r-\delta)h} \) if \( d > e^{(r-\delta)h} \), we do the opposite.
Example 10.1 pp. 317-318 provides an example.

**Arbitraging a mispriced option:**
if option overpriced, sell the option and buy a synthetic option (buying \( \frac{2}{3} \) shares and borrowing \( B \)).
if option underpriced, buy the option and sell a synthetic option (short-sell \( \frac{2}{3} \) shares and invest \( B \) in bonds).
see tables on pp 318 and 319 for illustration. Figure 10.2 pp 320 show the equivalence of the two scenarios (with only two points E and D considered and assume \( \delta = 0 \)).

**Risk-neutral pricing**
Let \( p^* = \frac{e^{(r-\delta)h} - d}{u-d} \). The option premium can be reexpressed as
\[
C = e^{-rh}[p^* C_u + (1 - p^*) C_d],
\]
which has the appearance of a discounted expected value. Note that \( p^* \) in general is not a probability. However, if we use \( p^* \) as the probability of an up move then the expected stock price equals the forward price
\[
p^* u S + (1 - p^*) d S = e^{(r-\delta)h} F_{t,t+h}.
\]
We call \( p^* \) the risk-neutral probability of an increase in the stock price.

**Construct a binomial tree**
If there is no uncertainty, the stock price in the future should be the forward price
\[
S_{t+h} = F_{t,t+h} = S_t e^{(r-\delta)h}.
\]
The stock price must rise at the risk-free rate less the dividend yield.

The question becomes how to gauge uncertainty. We link it to the annual standard deviation \( \sigma \), where the std. dev. for a period with length \( h \) will be \( \sigma \sqrt{h} \), often called volatility. We now model the stock price evolution as
\[
u S_t = F_{t,t+h} e^{\sigma \sqrt{h}}; \quad d S_t = F_{t,t+h} e^{-\sigma \sqrt{h}}.
\]
In other words, \( u = e^{(r-\delta)h + \sigma \sqrt{h}} \) and \( d = e^{(r-\delta)h - \sigma \sqrt{h}} \).

Another one-period example on page 322 and 323. using \( h = 1, \sigma = 0.3, S_0 = 41, r = 0.08, \delta = 0 \).

**10.2 Two or more binomial periods**
Figure 10. 4 on page 324 shows a two-period binomial tree. We work backwards.

**10.3 Put options**
similar to call options. the only difference is that at expiration, the option price is based on \( \max(0, K - S) \) instead of \( \max(0, S - K) \).
10.4 American Options

It is easy to check at each node whether early exercise is optimal, the binomial method is well-suited to valuing American options. The value of the option, if it is left unexercised, is given by the value of holding it for another period, if it is exercised, is given by $\max(0, S - K)$ for a call and $\max(0, K - S)$ if it is a put.

For an American put, the value of the option at a node is given by

$$P(S, K, t) = \max(K - S, e^{-rh}[P(uS, K, t + h)p^* + P(dS, K, t + h)(1 - p^*)])$$.

10.5 Options on other assets

Option on a stock index very much the same as analysis before.

Option on currencies where $\delta$ replaced by $rf$ the interest rate of the foreign currency.

Option on future contracts

The nodes are constructed as $u = e^{\sigma\sqrt{h}}$ and $d = e^{-\sigma\sqrt{h}}$. In each period a futures contract pays the change in the futures price, there is no investment required to enter a futures contract. The problem is to find the number of futures contracts, $\Delta$, and the lending, $B$, that replicates the option.

$$\Delta \times (dF - F) + e^{rh} \times B = C_d;$$

$$\Delta \times (uF - F) + e^{rh} \times B = C_u.$$

Therefore,

$$\Delta = (C_u - C_d)/[F(u - d)];$$

$$B = e^{-rh} (C_u \frac{1 - d}{u - d} + C_d \frac{u - 1}{u - d}),$$

while $\Delta$ tells us how many futures contracts to hold to hedge the option, the value of the option in this case is simply $B$.

Option on bonds

bonds differ from the assets we have been discussing in two important aspects:

1. the volatility of bonds decreases over time.
2. the interest rates vary over time.