Advertising Competition with Market Expansion for Finite Horizon Firms

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Journal of Industrial and Management Optimization, forthcoming
February 2005

\textbf{Abstract.} Firms that want to increase the sales of their brands through advertising have the choice of capturing market share from their competitors through brand advertising, or increasing primary demand for the category through generic advertising. In this paper, differential game theory is used to analyze the effects of the two types of advertising decisions made by firms offering a product in a dynamic duopoly. Each firm’s sales depend not only on its own and its competitor’s brand advertising strategies, but also on the generic advertising expenditures of the two firms. Closed-loop Nash equilibrium solutions are obtained for symmetric and asymmetric competitors in a finite-horizon setting. The analysis for the symmetric case results in explicit solutions, and numerical techniques are employed to solve the problem for asymmetric firms.

1 Introduction

There are two types of advertising available to a firm to increase the sales of its brand offering, namely, generic advertising which expands the market for the entire product category and brand advertising which increases the firm’s market share vis-à-vis other brands in the category. Brand advertising makes consumers aware that the brand is a potential substitute for the brand they currently use. It provides consumers with information about the brand’s characteristics, and thus helps differentiate the brand from competing brands and attract customers from the firm's competitors. Generic advertising generates new customers by targeting beliefs about product categories and not particular brands within the category.

In this paper, we determine how much firms should contribute towards generic advertising and how firm and competitive factors influence their contributions. Extant research on optimal advertising policies for competition is

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\textbf{Subject Classification:} 90B60, 49N70, 49N90, 49J20, 91A23

\textbf{Keywords:} Marketing; Advertising; Differential games; Optimal control; Dynamic duopoly

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(e.g., [15]) or dynamic models that do not explicitly consider generic advertising (e.g., [19, 5, 21, 23]). Thus, a contribution of this paper is to examine dynamic competition in brand and generic advertising. Since both advertising and sales are time-varying, we use dynamic optimization techniques and, in particular, differential game theory to analyze the situation [22, 6, 11].

We consider an advertising differential game between two finite-horizon firms. The assumption of a finite horizon allows us to consider time-varying parameters of the model, whereas an infinite-horizon analysis usually deals with stationary solutions in which the parameters are time-invariant. Since a finite-horizon analysis is not stationary, it provides additional insights about the end of the planning horizon. This is similar to the decline period of the product life cycle, which is a marketing concept that argues that brands have finite lives (Kotler [14]). Our analysis supports the notion that marketing activities are concentrated up front and there is almost no marketing at the end of the product life cycle, provided the salvage value is low. This is because the product is being harvested for whatever can be obtained. In addition, a finite-horizon analysis can incorporate a salvage value associated with the sales rate at the terminal time. It also facilitates numerical analysis when an explicit solution cannot be obtained.

Infinite horizon analysis of related dynamic duopoly models is available in Prasad and Sethi [17] and Bass et al. [1]. The finite-horizon analysis is significantly different and more complicated because the value function is not independent of the current time or the time remaining. As a result, numerical analysis is the norm in the literature. In this paper, however, we are able to proceed analytically to a significant extent by making use of the formulation in Sethi [21], which is amenable to closed-loop analysis.

The paper is organized as follows: Section 2 contains a review of the relevant literature. Section 3 presents the finite-horizon model. Section 4 deals with its analysis and presents the results for symmetric and asymmetric firms. Finally, Section 5 concludes with a summary and directions for future research.

2 Literature Review

There is a large literature in marketing about the effect of advertising on sales. Of this, the normative literature on the optimal control of advertising decisions has been surveyed by Sethi [20], Jorgensen [10], Erickson [7], and Jorgensen and Zaccour [11].

The earliest and commonly-used model of sales response to brand advertising is the Vidale-Wolfe model [24]. This model is given by

\[ \dot{S}(t) = \rho u(t)(M - S(t)) - \delta S(t), \quad S(0) = S_0, \]

where \( S(t) \) is the sales at time \( t \), \( u(t) \) is the advertising spending at time \( t \), \( \rho \) is the effectiveness of advertising, \( M \) is the saturation level of sales beyond which advertising has no effect, and \( \delta \) is a parameter that reflects the advertising
decay as a result of competition, obsolescence, and forgetting. Sethi [19] provides optimal advertising policies for the Vidale-Wolfe model.

To incorporate competitive effects into this model of advertising, researchers have used the Lanchester model of combat [13] to represent the market share dynamics and the competitive shifts due to the current investments in advertising by the two firms. The rate of change of market share in a Lanchester duopoly is written as

\[
\dot{x}_i(t) = \rho_i u_i(t)(1 - x_i(t)) - \rho_j u_j(t)x_i(t), \quad x_i(0) = x_{i0},
\]

where \(x_i(t)\) is the market share of firm \(i\) at time \(t\), \(u_i(t)\) is the brand advertising of firm \(i\), and \(\rho_i\) is the effectiveness of firm \(i\)'s brand advertising. This model is a competitive extension of the Vidale-Wolfe model of advertising in which firm \(i\) uses its advertising to capture firm \(j\)'s market share, and vice versa.

Sethi [21] developed a variant of (1) and used it to derive optimal advertising policies in a monopoly. Sorger [23] extended the Sethi [21] model to study brand advertising competition. This model is given by

\[
\dot{x}_i(t) = \rho_i u_i(t)\sqrt{1 - x_i(t)} - \rho_j u_j(t)\sqrt{x_i(t)}, \quad x_i(0) = x_{i0}.
\]

Sorger [23] discusses various desirable properties of this model and compares the model to other models of brand advertising. He derives explicit solutions for the optimal advertising expenditures so as to maximize each firm’s profit.

The effect of generic advertising on primary demand has been modeled in the agricultural economics literature [16, 12]. These empirical studies report good model fits and find a positive relationship between generic advertising and primary demand.

In marketing, Krishnamurthy [15] studied the relationship between generic and brand advertising for two cases – one in which firms contribute voluntarily to the generic advertising campaign, and another in which the government legislates funding for the campaign. However, this model is static and does not capture the dynamics of sales and the two types of advertising.

Fruchter [9] studied the dynamics of generic and brand advertising by proposing a model of oligopolistic advertising competition with market expansion due to competitive advertising. Firm’s profit function is given by

\[
V_i = \int_0^\infty e^{-rt}(m_iS_i(t) - u_i(t)^2)dt,
\]

where \(m_i\) is the margin and \(u_i(t)\) is firm \(i\)'s advertising. The equation of motion is specified as

\[
\dot{S}_i(t) = \rho_i u_i(t)(m(t) - S_i(t)) - S_i(t) \sum_{j=1, j \neq i}^n \rho_j u_j(t), \quad S_i(0) = S_{i0}.
\]
In the above model, both types of advertising are modeled using one advertising variable, so the differential effects of generic and brand advertising on sales are not captured.

Espinosa and Mariel [8] consider a dynamic model of oligopolistic advertising competition. The model is that of maximizing (4)

\[
\begin{align*}
\dot{S}_i(t) &= w_i(u_i(t) - u_j(t)) + z_i(u_i(t) + u_j(t)) + K_i - \delta_i S_i(t), \quad S_i(0) = S_{i0}, \\
\end{align*}
\]

where \(m_i\) is the gross margin, \(u_i(t)\) is firm \(i\)'s total advertising, and \(\delta_i\) is a decay term. As equation (6) illustrates, brand advertising affects the entire market, and not just the competitor’s portion, so there is no distinction between generic and brand advertising in this model. The authors obtain explicit solutions for some particular cases, but not for the full model as specified in (6).

3 Model

We consider a dynamic duopoly. Following the literature [7, 11], we assume that advertising is the dominant marketing mix variable in the product category and other marketing mix elements may be ignored. Since sales responds differently to generic and brand advertising, their effects have to be modeled separately. We start by modeling the effect of generic advertising on category demand. Since the industry demand rate \(\dot{Q}(t) = S_1(t) + S_2(t)\), the change in primary demand \(\dot{Q}(t)\) is equal to \(\dot{S}_1(t) + \dot{S}_2(t)\), where \(\dot{S}_i(t)\) is the rate of change of firm \(i\)'s sales. This is further specified as

\[
\begin{align*}
\dot{Q}(t) &= \dot{S}_1(t) + \dot{S}_2(t) = k_i(t)a_i(t) + k_j(t)a_j(t), \\
Q(0) &= Q_0, \quad i, j \in \{1, 2\}, \quad i \neq j, \\
\end{align*}
\]

where \(a_i(t)\) is the generic advertising of firm \(i\) at time \(t\) and \(k_i(t)\) is the effectiveness of firm \(i\)'s generic advertising.

The increase in the category demand as a result of generic advertising is shared unequally by the two firms. Let \(\theta_i(t) \in [0, 1]\), the ”allocation coefficient,” denote the proportion of the sales increase that is transferred to firm \(i\). The effect of generic advertising on firm \(i\)'s sales, denoted \(\dot{S}_{i,g}(t)\), is then

\[
\begin{align*}
\dot{S}_{i,g}(t) &= \theta_i(t)(k_i(t)a_i(t) + k_j(t)a_j(t)), \quad S_i(0) = S_{i0}. \\
\end{align*}
\]

We now model the effect of brand advertising on sales, denoted \(\dot{S}_{i,b}(t)\). This model is specified as

\[
\begin{align*}
\dot{S}_{i,b}(t) &= \rho_i(t)u_i(t)\sqrt{Q(t) - S_i(t) - \rho_j(t)u_j(t)\sqrt{S_i(t)}}, \quad S_i(0) = S_{i0}, \\
\end{align*}
\]

where \(u_i(t)\) is the brand advertising decision of firm \(i\) at time \(t\) and \(\rho_i(t)\) is the effectiveness of that advertising. Hence, the brand advertising model is based
on Sethi [21] and Sorger [23]. The total change in firm $i$’s sales is $\dot{S}_i(t) = \dot{S}_{i,g}(t) + \dot{S}_{i,b}(t)$. Adding equations (2) and (3), the total effect of generic and brand advertising on firm $i$’s sales rate is

$$\dot{S}_i(t) = \rho_i(t)u_i(t)\sqrt{Q(t) - S_i(t)} - \rho_j(t)u_j(t)\sqrt{S_j(t)} + \theta_i(t)(k_i(t)a_i(t) + k_j(t)a_j(t)), \quad S_i(0) = S_{i0},$$

where $S_{i0}$ is the initial sales of firm $i$. Rewriting the equations of motion for the two firms using $S_j(t) = Q(t) - S_i(t)$ yields

$$\dot{S}_1(t) = \rho_1(t)u_1(t)\sqrt{S_2(t)} - \rho_2(t)u_2(t)\sqrt{S_1(t)} + \theta(t)(k_1(t)a_1(t) + k_2(t)a_2(t)), \quad S_1(0) = S_{10},$$

$$\dot{S}_2(t) = \rho_2(t)u_2(t)\sqrt{S_1(t)} - \rho_1(t)u_1(t)\sqrt{S_2(t)} + (1 - \theta(t))(k_1(t)a_1(t) + k_2(t)a_2(t)), \quad S_2(0) = S_{20}.$$  

The two equations in (5-6) are intuitive in that the change in sales of one firm is given by the gain in sales due to its brand advertising $(\rho_iu_i(t)\sqrt{S_j(t)})$ minus the loss in sales due to the rival’s brand advertising $(\rho_ju_j(t)\sqrt{S_i(t)})$ plus the gain in sales due to market expansion $(\theta_i(k_1a_1(t) + k_2a_2(t)))$.

The control variables available to each firm are its generic and brand advertising decisions. Firm $i$’s profit maximization problem is

$$\max_{u_i(t), a_i(t)} V_{i0} \equiv V_i(S_{i0}, S_{j0}, 0) = \int_0^T e^{-\int_0^t r_i(\tau)d\tau} (m_i(t)S_i(t) - C(u_i(t), a_i(t), t))dt + e^{-\int_0^T r_i(\tau)d\tau} B_iS_i(T),$$

where $r_i(t)$ is the instantaneous discount rate of firm $i$, $m_i(t)$ is the per-unit margin of firm $i$, $T$ is the planning horizon, $B_i \geq 0$ is the per-unit salvage value of the sales rate of firm $i$ at time $T$, and $C(u_i(t), a_i(t), t)$ is the total advertising spending of firm $i$ at time $t$.

Firm $i$’s total advertising expense is specified as

$$C(u_i(t), a_i(t), t) = \frac{c_i(t)}{2}(a_i(t)^2 + u_i(t)^2),$$

where $c_i(t)$ is firm $i$’s advertising cost parameter. As in most of the literature, the cost of advertising is assumed to be quadratic (e.g., [7, 11]). Alternatively, one can use linear advertising costs and have advertising appear as a square root in the state equations.

The discounted profit maximization problems of the two firms can be rewritten as
\[
\max_{u_1(t), a_1(t)} V_1(S_{10}, S_{20}, 0) = \int_0^T e^{-\int_0^T r_1(\tau) d\tau} (m_1(t)S_1(t) - \frac{c_1(t)}{2}(a_1(t)^2 + u_1(t)^2)) dt \\
+ e^{-\int_0^T r_1(\tau) d\tau} B_1 S_1(T),
\]

\[(9)\]

\[
\max_{u_2(t), a_2(t)} V_2(S_{10}, S_{20}, 0) = \int_0^T e^{-\int_0^T r_2(\tau) d\tau} (m_2(t)S_2(t) - \frac{c_2(t)}{2}(a_2(t)^2 + u_2(t)^2)) dt \\
+ e^{-\int_0^T r_2(\tau) d\tau} B_2 S_2(T),
\]

\[(10)\]

s.t. \[
\dot{S}_1(t) = \rho_1(t)u_1(t)\sqrt{S_2(t)} - \rho_2(t)u_2(t)\sqrt{S_1(t)} \\
+ \theta(t)(k_1(t)a_1(t) + k_2(t)a_2(t)), \quad S_1(0) = S_{10},
\]

\[(11)\]

\[
\dot{S}_2(t) = \rho_2(t)u_2(t)\sqrt{S_1(t)} - \rho_1(t)u_1(t)\sqrt{S_2(t)} \\
+ (1 - \theta(t))(k_1(t)a_1(t) + k_2(t)a_2(t)), \quad S_2(0) = S_{20},
\]

\[(12)\]

where \(V_i\) is firm \(i\)'s profit function, also known as the value function.

4 Analysis and Results

The advertising differential game in (9-12) can be analyzed to yield either open-loop or closed-loop equilibria. In this paper, we adopt the closed-loop solution concept. This strategy better reflects the competitive dynamics of the two rivals over time. Moreover, a closed-loop equilibrium is more robust and is subgame perfect. In addition, Erickson [7] and Jorgensen and Zaccour [11] provide evidence that a closed-loop solution fits empirical data better than its open-loop counterpart.

The optimal advertising policies are given in Theorem 1. As is standard in the literature, we will suppress the time-dependence of various parameters and coefficients for expositional ease when no confusion arises.

Theorem 1 The differential game has a unique closed-loop Nash equilibrium solution for the two firms \(i\) and \(j\) with

a) the brand advertising decision given by

\[
u_i^* = \frac{\rho_i(t)}{c_i(t)}(\beta_i(t) - \gamma_i(t))\sqrt{S_i},\]

\[(1)\]
b) the generic advertising decision given by

\[ a_i^* = \frac{k_i(t)}{c_i(t)}(\theta_i \beta_i(t) + \theta_j \gamma_i(t)), \]

(2)

c) the value function \( V_i \) given by

\[ V_i \equiv V_i(S_i, S_j, t) = \alpha_i(t) + \beta_i(t)S_i + \gamma_i(t)S_j, \quad V_i(S_i, S_j, T) = B_i S_i(T), \]

(3)

where the coefficients \( \alpha_i(t), \beta_i(t), \gamma_i(t), \alpha_j(t), \beta_j(t), \) and \( \gamma_j(t) \) solve the simultaneous differential equations

\[ \dot{\alpha}_i = r_i \alpha_i - \frac{k_i^2}{2c_i} (\theta_i \beta_i + \theta_j \gamma_i)^2 - \frac{k_i^2}{c_j} (\theta_i \beta_i + \theta_j \gamma_i)(\theta_i \gamma_j + \theta_j \beta_j), \quad \alpha_i(T) = 0, \]

(4)

\[ \dot{\beta}_i = r_i \beta_i - m_i + \frac{\rho_i^2}{c_j} (\beta_i - \gamma_i)(\beta_j - \gamma_j), \quad \beta_i(T) = B_i, \]

(5)

\[ \dot{\gamma}_i = r_i \gamma_i - \frac{\rho_i^2}{2c_i} (\beta_i - \gamma_i)^2, \quad \gamma_i(T) = 0, \]

(6)

\[ \forall i, j \in \{1, 2\}, \quad i \neq j. \]

(All proofs in Appendix.)

From Theorem 1(a), each firm’s brand advertising is increasing in its competitor’s sales. This is intuitive because the smaller firm has a larger target of potential customers. The expression for the optimal generic advertising decision in Theorem 1(b), however, is relatively less straightforward to interpret, and further analysis is required.

For any explicit solution to be obtained, one would have to know the functional form of the time-varying model parameters, \( r_i(t), m_i(t), c_i(t), \rho_i(t), \theta_i(t), \) and \( k_i(t), i \in \{1, 2\} \). Since there is no agreement as to the form these parameters should take, we will assume for the remainder of the paper that the model parameters are constant. We will also assume the per-unit salvage value, \( B_i \), is zero for \( i \in \{1, 2\} \).

Let us first consider the analysis for symmetric firms.
4.1 Symmetric Firms

We start with symmetric firms maximizing their profits over a finite horizon. In this case, \( \alpha_i = \alpha_j = \alpha, \beta_i = \beta_j = \beta, \gamma_i = \gamma_j = \gamma, \ k_i = k_j = k, \ \rho_i = \rho_j = \rho, \ m_i = m_j = m, \ c_i = c_j = c, \ r_i = r_j = r, \) and \( \theta_i = \theta_j = \frac{1}{2}. \) Note that the initial market shares of the two firms can be different.

The optimal advertising policies are given in the following theorem:

Theorem 2 From Theorem 1, for a symmetric duopoly, the optimal brand and generic advertising decisions of firm \( i \) are given by

\[
\begin{align*}
u^*_i &= \frac{\rho(c(\beta(t) - \gamma(t)))}{e^{\beta(t) - \gamma(t)}} \sqrt{S_j}, \quad \alpha^*_i = \frac{k}{2c}(\beta(t) + \gamma(t)), \quad \text{where} \quad (7) \\

\beta(t) &= \frac{m}{3r}(1 - e^{-\frac{\rho^2}{c^2}(T-t)} + \frac{4}{1 + \frac{6\rho^2}{c^2} \coth\left(\frac{r}{2} (T-t) \sqrt{1 + \frac{6\rho^2}{c^2}}\right)}), \quad (8) \\

\gamma(t) &= \frac{m}{3r}(1 - e^{-\frac{\rho^2}{c^2}(T-t)} - \frac{2}{1 + \frac{6\rho^2}{c^2} \coth\left(\frac{r}{2} (T-t) \sqrt{1 + \frac{6\rho^2}{c^2}}\right)}), \quad (9)
\end{align*}
\]

The characteristics of the optimal closed-loop generic and brand advertising policies for symmetric firms, given in (7), are summarized in the following corollary:

Corollary 1. From Theorem 2, for symmetric firms in the finite-horizon case,

a) The optimal generic advertising of the two firms decreases with time;

b) The sales rate of the smaller firm (market share \( < \frac{1}{2} \)) increases with time.

Observation 1. From Theorem 2, for symmetric firms in the finite-horizon case,

a) For the larger firm, the ratio of the optimal generic advertising to the optimal brand advertising decreases with time;

b) The optimal brand advertising of the larger firm (market share \( > \frac{1}{2} \)) increases with time initially, but decreases to zero at the end of the planning horizon.
Note that, for both firms, generic and brand advertising decrease to zero at the end of the planning horizon. Corollary 1(a) states that most of the gains from generic advertising are felt in the beginning of the planning horizon. This is consistent with the belief that the goal of generic advertising during the introductory phase of the product life cycle is to educate consumers about the product category. An example in this context is the use of advertisements by Sirius and XM who are competing in the nascent market for satellite radio. Douglas Wilsterman, Sirius Satellite Radio’s VP of Marketing and Distribution, says, “You’ve got to do a little bit of both. You can’t just talk about yourself without people knowing what you represent in terms of a revolutionarily dynamic change” [2]. Along similar lines, Steve Cook, XM Satellite Radio’s VP of Sales and Marketing, adds, “XM’s ads will be about continuing to “grow the whole category pie,” rather than competing with Sirius” [3].

Corollary 1(b) states that the sales rate of the smaller firm increases with time. This is because the equilibrium sales of the two firms are equal. Note that, because of market expansion due to generic advertising by the two firms, the sales of the larger firm need not always decrease with time. If the market expansion effect dominates, the sales of both firms increase with time. In the absence of generic advertising, the sales of the larger firm decreases monotonically with time until it reaches the symmetric market share equilibrium.

As the smaller firm’s sales increase with time, it acts as an incentive for the larger firm to spend more on brand advertising because of the potential gain from capturing the rival’s sales. Therefore, the larger firm’s brand advertising increases initially. However, the zero salvage value assumption causes the larger firm’s brand advertising to decrease to zero toward the end of the planning horizon. This is part (b) of Observation 1. The result in part (a) follows from part (a) of Corollary 1 and part (b) of Observation 1.

Figure 1: Optimal Brand Advertising for Symmetric Firms: $S_{20} > S_{10}$
($r = 0.1$, $T = 6$, $m = 2$, $\rho = 0.7$, $c = 1.5$, $k = 1.2$, $S_{10} = 250$, $S_{20} = 400$)

The optimal trajectories of the larger firm’s sales and the smaller firm’s brand advertising are not easy to characterize as they are sensitive to the values
of the model parameters and the initial sales levels of the two firms. We now discuss the optimal policies for three broad cases.

Case 1: \( S_{i0} = S_{j0} \)

The fact that the sales of the two firms are equal implies that each firm starts off with, and remains at, the market share of 0.5. When the initial sales of the two firms are equal to zero, this reduces to the case of two firms launching a new product in the marketplace. The sales dynamics are affected by generic advertising only, so the optimal sales of the two firms increase with time. The assumption of symmetry, coupled with the same initial sales, means that the brand advertising levels of the two firms are equal. The optimal brand advertising of the two firms increases slowly with time, but goes to zero near the end of the planning horizon. As a result, the ratio of the optimal generic advertising to the optimal brand advertising decreases with time.

Case 2: \( S_{j0} > S_{i0} \)

This case is depicted in Figure 1. If the initial sales rate of firm \( j \) is higher than that of firm \( i \), the optimal sales rate of firm \( i \) increases with time. Firm \( j \)'s sales, on the other hand, decrease initially as the difference in the sales of the two firms decreases, later increase as a result of market expansion. The optimal brand advertising of firm \( j \) increases initially due to the increased potential to capture firm \( i \)'s sales, but decreases to zero toward the end of the planning horizon. The optimal brand advertising of firm \( i \) decreases with time and goes to zero at time \( T \). If the effectiveness of generic advertising is high, i.e., the market expansion effect dominates, the smaller firm’s brand advertising need not always decrease with time. If, on the other hand, brand advertising is highly effective, i.e., the sales expansion effect is negligible, the firms reach equal market shares quickly. Finally, if the effectiveness of brand advertising is negligible, the sales of the two firms increase monotonically with time. In this case, the optimal brand advertising of the smaller firm decreases with time, while that of the larger firm increases initially, but later goes to zero.

Case 3: \( S_{j0} >> S_{i0} \)

The context here is a “near-monopoly” in which firm \( j \) controls most of the market. In this case, the sales of the smaller firm, \( S_i(t) \), is increases monotonically with time, while \( S_j(t) \) is decreases monotonically with time. Therefore, the optimal brand advertising of firm \( i \) is very high initially because of the potential to gain sales from firm \( j \), but as firm \( i \)'s sales increase, its optimal brand advertising decreases because the potential to usurp sales from firm \( j \) is not as high as it was initially. Therefore, firm \( i \)'s optimal brand advertising decreases with time, and firm \( j \)'s correspondingly increases with time, before going to zero toward the end of the planning horizon. The ratio of the optimal generic advertising to the optimal brand advertising of firm \( i \) increases initially, but later decreases with time. It should be noted, however, that if the effectiveness of generic advertising is very high, i.e., when the market expansion effect dominates, the impact of the huge difference in initial sales is significantly reduced. For an illustration, see Figure 2.
We now compare the two extreme cases of pure market expansion (the effectiveness of generic advertising is much greater than that of brand advertising) and pure market share competition (the effectiveness of brand advertising is much greater than that of generic advertising). In the former case, symmetry implies that both firms gain equally from market expansion. Therefore, while the smaller firm gains from market expansion, it can never reach the sales level of the larger firm because the larger firm’s sales also increase by the same amount. The percentage increase in sales is greater for the smaller firm, so its market share increases a little each period, but the equilibrium of equal market shares is never reached because the difference in the initial sales levels can never be made up. However, in the case of pure market share competition, where the context is a mature market, the smaller firm gains sales at the expense of the larger firm, resulting in equal market shares in equilibrium.

Figure 2: Optimal Brand Advertising for Symmetric Firms: $S_{20} >> S_{10}$
($r = 0.1, T = 6, m = 2, \rho = 0.7, c = 1.5, k = 1.2, S_{10} = 50, S_{20} = 600$)

The comparative statics for the parameters on the variables of interest are given in Table 1. Since we have assumed symmetry, $u^*_2$, $a^*_2$, and $V_2$ have the same comparative statics as $u^*_1$, $a^*_1$, and $V_1$, respectively.

The results in Table 1 indicate that an increase in the effectiveness of brand advertising, $\rho$, increases the optimal brand advertising of the firm. This is because an increase in $\rho$ offers a greater incentive for the firm to spend more money on brand advertising. An increase in the brand advertising level translates into a corresponding decrease in generic advertising. In addition, an increase in $\rho$ decreases the firm’s value function. Next, consider the effect of an increase in the advertising cost parameter, $c$. An increase in $c$ decreases the optimal generic and brand advertising. This make sense economically. However, the firm’s value function increases as $c$ increases. An increase in the gross margin, $m$, increases the optimal advertising levels because it results in increased revenue if advertising is increased. Although some of this advertising might be wasted, it can also increase the size of the pie, and, hence, increase the value function.
As the discount rate, \( r \), increases, the optimal advertising levels decrease. The discount rate is a measure of the inherent risk in the industry, so a high value of \( r \) would imply less weight on future profits, resulting in decreasing optimal advertising and profit. In the limiting case, when the discount rate is extremely high, the firm acts myopically by trying to maximize only current-period profits.

As the effectiveness of generic advertising, \( k \), increases, the optimal generic advertising also increases. However, brand advertising is unaffected by \( k \) because \( k \) increases the sales of both firms without greatly affecting market shares. Finally, computing the effect of a wider planning horizon, \( T \), on the key variables, we find that as \( T \) increases, the value function parameters also increase, resulting in higher generic and brand advertising levels.

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Legend: ↑ increase; ↓ decrease; = unchanged

In the next section, we consider the case of asymmetric firms.

### 4.2 Asymmetric Firms

For asymmetric firms, one has to solve the following six simultaneous differential equations to obtain the optimal generic and brand advertising decisions:

\[
\dot{\alpha}_1 - r_1 \alpha_1 + \frac{k_2^2}{2c_1} (\theta \beta_1 + (1-\theta) \gamma_1)^2 + \frac{k_2^2}{c_2} (\theta \beta_1 + (1-\theta) \gamma_1)(\theta \gamma_2 + (1-\theta) \beta_2) = 0, \quad (10)
\]

\[
\dot{\beta}_1 - r_1 \beta_1 + m_1 - \frac{\rho_2^2}{c_2} (\beta_1 - \gamma_1)(\beta_2 - \gamma_2) = 0, \quad (11)
\]

\[
\dot{\gamma}_1 - r_1 \gamma_1 + \frac{\rho_1^2}{2c_1} (\beta_1 - \gamma_1)^2 = 0, \quad (12)
\]

\[
\dot{\alpha}_2 - r_2 \alpha_2 + \frac{k_2^2}{2c_2} (\theta \gamma_2 + (1-\theta) \beta_2)^2 + \frac{k_1^2}{c_1} (\theta \beta_1 + (1-\theta) \gamma_1)(\theta \gamma_2 + (1-\theta) \beta_2) = 0, \quad (13)
\]

\[
\dot{\beta}_2 - r_2 \beta_2 + m_2 - \frac{\rho_1^2}{c_1} (\beta_1 - \gamma_1)(\beta_2 - \gamma_2) = 0, \quad (14)
\]
\[ \gamma_2 - r_2 \gamma_2 + \frac{\rho_2^2}{2c_2} (\beta_2 - \gamma_2)^2 = 0. \] (15)

Since this system of differential equations cannot be solved in closed-form, we will have to use numerical methods to solve for \( \alpha_1(t) \), \( \beta_1(t) \), \( \gamma_1(t) \), \( \alpha_2(t) \), \( \beta_2(t) \), and \( \gamma_2(t) \).

The solutions to the system of simultaneous differential equations in (10-15) are obtained using Matlab v6.5. Since these equations do not involve the state variables, a two-stage solution procedure is used. In the first stage, the paths of \( \alpha_i(t) \), \( \beta_i(t) \), and \( \gamma_i(t) \) are numerically determined for different values of the model parameters, using the boundary condition \( \alpha_i(T) = \beta_i(T) = \gamma_i(T) = 0 \). In the next stage, we substitute the solutions for \( \alpha_i(t) \), \( \beta_i(t) \), and \( \gamma_i(t) \) and the optimal advertising decisions in (1-2) and solve the resulting system of differential equations to obtain the optimal state trajectories, \( S_1(t) \) and \( S_2(t) \). We can then substitute the solutions for \( \alpha_1(t) \), \( \beta_1(t) \), \( \gamma_1(t) \), \( \alpha_2(t) \), \( \beta_2(t) \), \( \gamma_2(t) \), \( S_1(t) \), and \( S_2(t) \) into equations (1-2) to obtain the optimal advertising decisions for the two firms.

Figure 3: Optimal Brand Advertising for Asymmetric Firms: \( S_{20} = S_{10} \)
\( (r_1 = r_2 = 0.1, \, T = 6, \, m_1 = m_2 = 2, \, \rho_1 = 0.7, \, \rho_2 = 0.8, \, c_1 = c_2 = 1.5, \, k_1 = k_2 = 1.2, \, \theta = 0.5, \, S_{10} = 325, \, S_{20} = 325) \)

The optimal brand advertising decisions for three representative cases are shown in Figures 3, 4, and 5. These are the asymmetric counterparts of the three cases for symmetric firms, discussed in the previous section. For tractability, we assume asymmetry only in the brand advertising effectiveness of the two firms. Define the “stronger” firm as the one with higher brand advertising effectiveness, and the “weaker” firm as that with lower brand advertising effectiveness. When the firms have equal initial sales, and differ only in their brand advertising effectiveness, this difference in advertising effectiveness directly translates into higher initial brand advertising for the stronger firm (Figure 3).
If the stronger firm has higher initial sales, the difference in the brand advertising levels of the two firms is not as high as it was earlier (Figure 4). This is because the weaker firm now has a larger target of potential customers.

Figure 4: Optimal Brand Advertising for Asymmetric Firms: \( S_{20} > S_{10} \)
\( (r_1 = r_2 = 0.1, \ T = 6, \ m_1 = m_2 = 2, \ \rho_1 = 0.7, \ \rho_2 = 0.8, \ c_1 = c_2 = 1.5, \ k_1 = k_2 = 1.2, \ \theta = 0.5, \ S_{10} = 250, \ S_{20} = 400) \)

Finally, if the difference in initial sales is substantial, the initial brand advertising of the weaker firm is much higher than that of its rival (Figure 5).

Figure 5: Optimal Brand Advertising for Asymmetric Firms: \( S_{20} >> S_{10} \)
\( (r_1 = r_2 = 0.1, \ T = 6, \ m_1 = m_2 = 2, \ \rho_1 = 0.7, \ \rho_2 = 0.8, \ c_1 = c_2 = 1.5, \ k_1 = k_2 = 1.2, \ \theta = 0.5, \ S_{10} = 50, \ S_{20} = 600) \)

Note that, because of the salvage value assumption, the brand advertising goes to zero at time \( T \). In summary, the optimal brand advertising strategies for asymmetric firms are very similar to those for symmetric firms. Therefore,
the numerical analysis provides support for the robustness of the results from the symmetric case.

5 Conclusions

A firm that wants to increase the sales of its brand can use generic advertising to expand the sales of the entire industry or brand advertising to increase its market share relative to competitors. The benefits from generic advertising differ from that of brand advertising in that the benefits of the former are shared by all the firms in the marketplace regardless of which firm contributed. The success of the firm’s overall promotion plan will depend on the optimal allocation of funds to generic and brand advertising and a thorough understanding of the nature of the relationship between the two.

In this paper, we consider market expansion due to generic advertising, and derive the closed-loop Nash equilibrium strategies for a dynamic duopoly in which firms make generic and brand advertising decisions in a finite-horizon setting. Finite-horizon problems are quite complicated and the emphasis here was on solvability. Explicit solutions were obtained for symmetric competitors, and numerical methods were used to solve the problem for asymmetric firms. The effects of the model parameters on the optimal advertising policies and profits were computed.

Every finite-horizon problem has a beginning game, a middle game, and an end game, provided the horizon is sufficiently large. If the horizon is not large enough, there may be no middle game. The initial conditions, for example, have a more important effect on optimal trajectories over the time frame. Consequently, we considered three representative initial conditions where the firms have different initial sales. In addition, we chose a salvage value of zero, which is a fairly standard assumption when no information on the terminal value of the firm is available.

The analysis reveals that the optimal generic advertising decreases monotonically with time (Theorem 1a). That is, generic advertising plays an important role during the introduction stage of the product life cycle. The optimal brand advertising can either increase or decrease with time depending on the model parameters. We dealt with three broad cases, and characterized the optimal trajectories in each case. Normative results are that the firm’s optimal advertising decisions should increase with its gross margin and decrease with its discount rate (Table 1).

There are several avenues for future research. Our analysis has concentrated on a specific model, but hopefully this will inspire researchers to generalize the model and analysis. In addition to advertising, pricing is an important decision variable in many markets and should be incorporated in an extended model. One may also extend the model to study advertising competition in an oligopoly along the lines of Fruchter [9] and Prasad and Sethi [18]. Finally, marketing insights depend upon how given parameters vary over the finite horizon. Since
there is no particular agreement as to how parameters may vary, one should try to estimate this empirically.
A Proofs

Proof of Theorem 1. The Hamilton-Jacobi-Bellman (HJB) equation for firm $i$ is

$$r_i V_i - \frac{\partial V_i}{\partial t} = \max_{u_i, a_i} \left\{ m_i S_i - \frac{c_j}{2} (a_i^2 + u_i^2) + \frac{\partial V_i}{\partial S_j} (\rho_i u_j \sqrt{S_j} - \rho_j u_i \sqrt{S_i} + \theta_i (k_i a_i + k_j a_j)) + \frac{\partial V_i}{\partial S_j} (\rho_j u_j \sqrt{S_j} - \rho_i u_i \sqrt{S_i} + \theta_j (k_i a_i + k_j a_j)). \right\}$$

From this, the first-order conditions for $u_i$ and $a_i$ yield, respectively,

$$u_i^* = \frac{\rho_i}{c_i} (\frac{\partial V_i}{\partial S_i} - \frac{\partial V_i}{\partial S_j}) \sqrt{S_j}, \quad a_i^* = \frac{k_i}{c_i} (\theta_i \frac{\partial V_i}{\partial S_i} + \theta_j \frac{\partial V_i}{\partial S_j}).$$

Substituting (2) into the HJB equation in (1) yields

$$r_i V_i = m_i S_i + \frac{k_i^2}{2c_i} (\theta_i \frac{\partial V_i}{\partial S_i} + \theta_j \frac{\partial V_i}{\partial S_j})^2 + \frac{\rho_i^2}{2c_i} (\frac{\partial V_i}{\partial S_i} - \frac{\partial V_i}{\partial S_j})^2 S_j + \frac{\partial V_i}{\partial S_j} - \frac{\rho_i^2}{c_j} (\frac{\partial V_i}{\partial S_i} - \frac{\partial V_i}{\partial S_j}) (\frac{\partial V_i}{\partial S_j} - \frac{\partial V_i}{\partial S_j}) S_i + \frac{k_i^2}{c_j} (\theta_i \frac{\partial V_i}{\partial S_i} + \theta_j \frac{\partial V_i}{\partial S_j}) (\theta_i \frac{\partial V_i}{\partial S_i} + \theta_j \frac{\partial V_i}{\partial S_j}).$$

Observe that a linear value function $V_i = \alpha_i + \beta_i S_i + \gamma_i S_j$ satisfies (3). The optimal brand and generic advertising decisions can now be rewritten as

$$u_i^* = \frac{\rho_i(t)}{c_i(t)} (\beta_i(t) - \gamma_i(t)) \sqrt{S_j}, \quad a_i^* = \frac{k_i(t)}{c_i(t)} (\theta_i \beta_i(t) + \theta_j \gamma_i(t)).$$

Substituting the linear value function and (4) into (3) and simplifying, we have

$$r_i \alpha_i + r_i \beta_i S_i + r_i \gamma_i S_j = m_i S_i + \frac{k_i^2}{2c_i} (\theta_i \beta_i + \theta_j \gamma_i)^2 + \frac{\rho_i^2}{2c_i} (\beta_i - \gamma_i)^2 S_j + \alpha_i + \beta_i S_i + \gamma_i S_j - \frac{k_i^2}{c_j} (\beta_i - \gamma_i)(\beta_j - \gamma_j) S_i + \frac{k_j^2}{c_j} (\theta_i \beta_i + \theta_j \gamma_i)(\theta_i \gamma_j + \theta_j \beta_j).$$

Equating the coefficients of $S_i$, $S_j$, and the constant in equation (5) results in the following differential equations to solve for $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$, $\alpha_j(t)$, $\beta_j(t)$, and $\gamma_j(t)$:

$$\dot{\alpha}_i = r_i \alpha_i - \frac{k_i^2}{2c_i} (\theta_i \beta_i + \theta_j \gamma_i)^2 - \frac{k_j^2}{c_j} (\theta_i \beta_i + \theta_j \gamma_i)(\theta_i \gamma_j + \theta_j \beta_j),$$

$$\dot{\beta}_i = r_i \beta_i - m_i + \frac{\rho_i^2}{c_j} (\beta_i - \gamma_i)(\beta_j - \gamma_j),$$

and $\dot{\gamma}_i(t)$ follows from equation (5).
\[ \dot{\gamma}_i = r_i \gamma_i - \frac{\rho_i^2}{2c_i} (\beta_i - \gamma_i)^2. \]  

(8)

The salvage value assumption provides the boundary conditions, i.e., \( V_i(T) = B_i S_i(T) \), so \( \alpha_i(T) = \gamma_i(T) = 0 \), and \( \beta_i(T) = B_i \). ■

**Proof of Theorem 2.** For symmetric firms, one has to solve the following system of differential equations:

\[ \dot{\alpha} = r \alpha - \frac{3k^2}{8c} (\beta + \gamma)^2, \quad \alpha(T) = 0, \]  

(9)

\[ \dot{\beta} = r \beta - m + \frac{\rho^2}{c} (\beta - \gamma)^2, \quad \beta(T) = 0, \]  

(10)

\[ \dot{\gamma} = r \gamma - \frac{\rho^2}{2c} (\beta - \gamma)^2, \quad \gamma(T) = 0. \]  

(11)

To solve these equations, multiply (11) by 2, add to (10), and let \( q = \beta + 2\gamma \). This yields

\[ \dot{q} - rq + m = 0, \quad q(T) = 0, \]  

the solution of which is

\[ q(t) = \frac{m}{r} (1 - e^{-r(T-t)}) \Rightarrow \beta = \frac{m}{r} (1 - e^{-r(T-t)}) - 2\gamma. \]  

(13)

Substituting this into (11) gives

\[ \dot{\gamma} - r \gamma + \frac{\rho^2}{2c} \left( \frac{m}{r} (1 - e^{-r(T-t)}) - 3\gamma \right)^2 = 0, \quad \gamma(T) = 0. \]  

(14)

From (14), one can obtain \( \gamma(t) \) to be

\[ \gamma(t) = \frac{m}{3r} (1 - e^{-r(T-t)}) - \frac{2}{1 + \sqrt{1 + \frac{6m\rho^2}{cr^2} \coth \left( \frac{r}{2} (T - t) \sqrt{1 + \frac{6m\rho^2}{cr^2}} \right)}}. \]  

(15)

After substituting (15) into (10), solving the resulting differential equation yields the following solution for \( \beta(t) \):

\[ \beta(t) = \frac{m}{3r} (1 - e^{-r(T-t)}) + \frac{4}{1 + \sqrt{1 + \frac{6m\rho^2}{cr^2} \coth \left( \frac{r}{2} (T - t) \sqrt{1 + \frac{6m\rho^2}{cr^2}} \right)}}. \]  

(16)

One can now substitute (15-16) into (9) and solve for \( \alpha(t) \).

Notice in equations (15-16) that \( \beta(t) \) and \( \gamma(t) \) are both non-negative, and \( \beta(t) \geq \gamma(t) \forall t \leq T \), resulting in non-negative values for the advertising controls.
In equation (16), both $e^{-r(T-t)}$ and $\coth\left(\frac{r}{2}(T-t)\sqrt{1 + \frac{6m\rho^2}{cr^2}}\right)$ increase with time. Therefore, $\beta(t)$ decreases with time $\forall t \leq T$. Moreover, from (15-16), $\beta(t)$ decreases faster with time than $\gamma(t)$. 

For $\gamma(t)$, we write (15) as

$$\gamma(t) = \frac{m}{3r}(1 - e^{-r(T-t)} - 2 \frac{1}{1 + y \coth\left(\frac{r}{2}(T-t)\right)}),$$

(17)

where $y \equiv \sqrt{1 + \frac{6m\rho^2}{cr^2}}$. Note that $y \geq 1$. We can rewrite (17) as

$$\gamma(t) = \frac{m}{3r}(1 - e^{-r(T-t)} - 2 \frac{1 + e^{-ry(T-t)}}{1 - e^{-ry(T-t)}y}).$$

(18)

Taking the derivative of (18) with respect to time and simplifying yields

$$\dot{\gamma}(t) = \frac{m}{3} e^{-r(T-t)}(-1 + 4y^2 e^{-3y(T-t)}(y - 1) e^{-ry(T-t)} \frac{1 + e^{-ry(T-t)}}{2}).$$

(19)

Equation (19) can be rewritten as

$$\dot{\gamma}(t) = \frac{m}{3} e^{-r(T-t)}(-1 + 4y^2 e^{-3y(T-t)}(y - 1) e^{-ry(T-t)} \frac{1 + e^{-ry(T-t)}}{2}).$$

(20)

It can be seen that, $\forall r \geq 0$, $y \geq 1$, and $t \leq T$,

$$\frac{4y^2 e^{-3y(T-t)}(y - 1) e^{-ry(T-t)}}{(y + 1 + (y - 1) e^{-ry(T-t)})^2} \leq 1.$$ 

(21)

Therefore, from (20-21), $\gamma(t)$ is also decreasing with time. Moreover, $\beta(t) > \gamma(t) > 0$ $\forall t < T$, so the advertising controls in equation (7) are positive.

**Proof of Corollary 1.** Firm $i$’s optimal generic advertising is given by

$$a_i^*(S_i, S_j, t) = \frac{k}{2c}(\beta(t) + \gamma(t)),$$

(22)

which is directly proportional to $\beta(t) + \gamma(t)$. Since both $\beta(t)$ and $\gamma(t)$ decrease with time, $\beta(t) + \gamma(t)$ also decreases with time. Therefore, the optimal generic advertising decreases with time (Corollary 1(a)).

The sales equations can now be written as

$$\dot{S}_1 = \frac{\rho^2}{c} (\beta - \gamma)(S_2 - S_1) + \frac{k^2}{2c} (\beta + \gamma), \ S_1(0) = S_{10},$$

(23)

$$\dot{S}_2 = \frac{\rho^2}{c} (\beta - \gamma)(S_1 - S_2) + \frac{k^2}{2c} (\beta + \gamma), \ S_2(0) = S_{20}.$$ 

(24)
If firm 1 is the smaller firm (market share < \( \frac{1}{2} \)), one can see from (23) that \( \dot{S}_1 > 0 \), so the smaller firm’s sales increases with time (Corollary 1(b)). However, the result for the larger firm depends on the relative magnitudes of market share attrition and market expansion.

**Proof of Observation 1.** From (15-16), we have

\[
\beta(t) - \gamma(t) = \frac{2m}{r} \left( \frac{1}{1 + \sqrt{1 + \frac{6m\rho^2}{cr^2}} \coth\left( \frac{r}{2}(T-t) \sqrt{1 + \frac{6m\rho^2}{cr^2}} \right)} \right), (25)
\]

which also decreases with time. Moreover, \( \beta(t) + \gamma(t) \) decreases faster with time than \( \beta(t) - \gamma(t) \), therefore the optimal generic advertising decreases faster with time than its optimal brand advertising (Observation 1(a)).

The result for the larger firm’s brand advertising can be obtained by using the result from Corollary 1 in conjunction with the expression for the optimal brand advertising decision. The larger firm’s brand advertising is given by

\[
u_2^* = \frac{\rho}{c} \left( \beta(t) - \gamma(t) \right) \sqrt{S_1}. (26)
\]

Numerical simulation shows that the combination \( \sqrt{S_1} \) increases with time initially as the increase in \( S_1 \) more than compensates for the decrease in \( \beta(t) - \gamma(t) \). However, as we approach the end of the horizon, the decrease in \( \beta(t) - \gamma(t) \) dominates, resulting in decreasing brand advertising in later periods (Observation 1(b)).

From (23-24), one can solve for \( S_1(t) \) and \( S_2(t) \) as follows:

\[
\dot{S}_1 + \dot{S}_2 = \frac{k^2}{c} (\beta + \gamma), \quad S_1(0) = S_{10}, \quad S_2(0) = S_{20}, (26)
\]

\[
\dot{S}_1 - \dot{S}_2 = \frac{2\rho^2}{c} (\beta - \gamma)(S_2 - S_1), \quad S_1(0) = S_{10}, \quad S_2(0) = S_{20}. (27)
\]

Solving the system of differential equations in (26-27) yields

\[
S_1(t) + S_2(t) = S_{10} + S_{20} + \frac{k^2}{c} \int_0^t (\beta(\tau) + \gamma(\tau)) d\tau, (28)
\]

\[
S_1(t) - S_2(t) = (S_{10} - S_{20}) e^{-\frac{4\rho^2}{c} \int_0^t (\beta(\tau) - \gamma(\tau)) d\tau}, (29)
\]

Solving (28-29) by substituting (15-16), using \( y \equiv \sqrt{1 + \frac{6m\rho^2}{cr^2}} \), simplifying, and integrating results in
\[ S_1(t) + S_2(t) = S_{10} + S_{20} + \frac{2mk^2}{3cr^2} \left( \frac{y + 2}{y + 1} \right) t \]
\[ - \frac{2}{y^2 - 1} \ln \left( \frac{y + 1 + (y - 1)e^{-ry(T-t)}}{y + 1 + (y - 1)e^{-ryT}} \right) + e^{-rT} - e^{-r(T-t)}, \]

(30)

\[ S_1(t) - S_2(t) = (S_{10} - S_{20}) e^{-\frac{2}{3} \left( 2 \ln \left( \frac{y - 1 + (y + 1)e^{ryT}}{y - 1 + (y + 1)e^{ry(T-t)}} \right) - (y + 1)rt \right)}. \]

(31)

From (30-31), we get

\[ S_1(t) = \frac{1}{2} \left( \frac{2mk^2}{3cr^2} \right) \left( \frac{y + 2}{y + 1} \right) t - \frac{2}{y^2 - 1} \ln \left( \frac{y + 1 + (y - 1)e^{-ry(T-t)}}{y + 1 + (y - 1)e^{-ryT}} \right) + e^{-rT}
\]
\[ - e^{-r(T-t)} \right) + S_{10} + S_{20} + (S_{10} - S_{20}) e^{-\frac{2}{3} \left( 2 \ln \left( \frac{y - 1 + (y + 1)e^{ryT}}{y - 1 + (y + 1)e^{ry(T-t)}} \right) - (y + 1)rt \right)}, \]

(32)

\[ S_2(t) = \frac{1}{2} \left( \frac{2mk^2}{3cr^2} \right) \left( \frac{y + 2}{y + 1} \right) t - \frac{2}{y^2 - 1} \ln \left( \frac{y + 1 + (y - 1)e^{-ry(T-t)}}{y + 1 + (y - 1)e^{-ryT}} \right) + e^{-rT}
\]
\[ - e^{-r(T-t)} \right) + S_{10} + S_{20} + (S_{20} - S_{10}) e^{-\frac{2}{3} \left( 2 \ln \left( \frac{y - 1 + (y + 1)e^{ryT}}{y - 1 + (y + 1)e^{ry(T-t)}} \right) - (y + 1)rt \right)}. \]

(33)

**Acknowledgement.** The authors thank Ram Rao, B. P. S. Murthi, Ernan Haruvy, and seminar participants at the University of Texas at Dallas, the University of California, Berkeley, and the University of California, Riverside, and three anonymous Reviewers for their valuable suggestions.
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