Ch.5 Feedback Control System

Objectives
• Covered: Math. Modeling & Transfer Function
• New: include control system characteristics
  1. Sensitivity to model uncertainties
  2. Steady-state errors \( (t \to \infty) : Y(\infty), E(\infty) \)
  3. Transient response characteristics to input test signals
  4. Disturbance rejection

→ Find TF of a feedback control system
**Closed-loop control**

- Has ability to reduce system sensitivity
- If $G(s)H(s)>>1$ for all complex freq. of interest, then:

$$Y(s) = \frac{G}{1 + G \cdot H} R \rightarrow \frac{G}{G \cdot H} R = \frac{1}{H} R$$

If $H = 1$, $Y(s) = R(s)$  the desired result

- By increasing the gain of $G(s)H(s)$ it reduces the effect of $G(s)$ on the input -> variation of the parameters of the process, $G(s)$, is reduced *(advantage of a feedback system)*
- But, making $G(s)H(s)>>1$ can lead to highly oscillatory & even unstable response
When Process, $G(s)$, is changed

\[ \Delta Y(s) = \Delta G(s)R(s) \]

\[ R(s) \rightarrow \boxed{G(s)} \rightarrow Y(s) \]

\[ \begin{align*}
Y(s) + \Delta Y(s) &= \frac{G(s) + \Delta G(s)}{1 + (G(s) + \Delta G(s))H(s)}R(s) \\
Y(s) &= \frac{G}{1 + G \cdot H}R
\end{align*} \]

Then the change in the output is

\[ \Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s) + \Delta GH(s))(1 + GH(s))}R(s). \]

When $GH \gg \Delta GH(s)$, as is often the case, we have

\[ \Delta Y(s) = \frac{\Delta G(s)}{[1 + GH(s)]^2}R(s). \]

Original T. F.

\[ \frac{G}{1 + G \cdot H} \]

Change of the output is reduced by $[1 + GH]$.
<System Sensitivity>

\[ S_T^G = \frac{\text{Ratio of } \% \text{ change in sys T.F.}}{\text{Ratio of } \% \text{ change in “Process” T.F.}} \]

\[ \frac{\Delta T}{\Delta G} = \frac{\partial \ln T}{\partial \ln G} = \frac{T}{G} \frac{\partial T}{\partial G} \]

**Open-loop**

\[ \Delta Y(s) = \Delta G(s)R(s), \quad \Delta T(s) = \frac{\Delta Y(s)}{R(s)} = \Delta G(s) \]

**Closed-loop**

\[ T(s) = \frac{G}{1 + GH} , \quad \frac{\partial T}{\partial G} = \frac{(1 + GH) - G(H)}{(1 + GH)^2} = \frac{1}{(1 + GH)^2} \]

\[ S_T^G = \frac{\partial T}{\partial G} \cdot \frac{G}{T} = \frac{1}{(1 + GH)^2} \cdot \frac{G}{1 + GH} = \frac{1}{(1 + GH)^2} \]

→ Reduced \( S_T^G \) below that of the open-loop sys by increasing \( G*H \) (>>1.0).

* if \( GH \gg 1.0 \) → \( S_T^G = -1 \)

Feedback components should not be varied with environmental changes → change in \( H(s) \) directly effect output response
< Disturbance in a system >

By superposition

\[
C(s) = \frac{K_\tau G_1 G_2}{1 + K_\tau G_1 G_2 H} M(s) + \frac{G_2}{1 + K_\tau G_1 G_2 H} D \tau(s) = G_p(s) M(s) + G_d(s) D \tau(s)
\]

State Variable Model

\[
\dot{X} = Ax + Bu \quad u = \begin{bmatrix} m(t) \\ d(t) \end{bmatrix} \\
y = Cx \\
\]

Then T.F. \( G(s) = C[sI-A]^{-1}B \)

\[
= \begin{bmatrix} G_p(s) & G_d(s) \end{bmatrix}
\]

T.F. from the control Input \( M(s) \) to the output

T.F. from \( D(s) \) to the output
Disturbance in a closed-loop system

\[
C(s) = \left( \frac{G_c G_p}{1 + G_c G_p H} \right) R(s) + \left( \frac{G_d}{1 + G_c G_p H} \right) D(s)
\]

\[
= T(s) R(s) + T_d(s) D(s)
\]

- The loop gain \( G_c \cdot G_p \cdot H \) must be made large to reduce the system sensitivity.

\[
T_d = \frac{G_d}{1 + G_c G_p H} = \frac{G_d}{G_c G_p H}
\]

Reducing Disturbance
1. Reduce the gain \( G_d(s) \)
2. Increase the loop gain \( G_c \cdot G_p \cdot H \) (Choice of \( G_c \))
3. Reduce the disturbance \( d(t) \)
4. Feed forward method if the disturbance can be measured
**Example:** Feed forward method

\[
T = \frac{\sum P_k \Delta_k}{\Delta}
\]

**T.F of the disturbance**

\[
T_d = \frac{G_d}{1 + G_c G_p H} + \frac{-G_{cd} G_c G_p}{1 + G_c G_p H}
\]

if \( G_{cd} \) is selected to make \( T_d = 0 \)

\[
G_d - G_{cd} G_c G_p = 0
\]

\[
\therefore G_{cd} = G_c \frac{G_d}{G_p}
\]
**Steady State Error:** Error after the transient response has decayed \( t \to \infty \)

**Open Loop**

\[ R(s) \rightarrow G(s) \rightarrow Y(s) \]

Error \( E_0(s) = R(s) - Y(s) = R(s) - G(s)R(s) = R(s) [1- G(s)] \)

**Closed Loop**

\[
E_C(s) = \frac{1}{1+G(s)} R(s)
\]

- The final value theorem \[ \lim_{t \to 0} e(t) = \lim_{s \to 0} sE(s) \]
- Error for unit step input \[ u(t) \to \frac{1}{s} \quad t \geq 0 \]

1) Open Loop case

\[ e_0(\infty) = \lim_{s \to 0} s[E(s)]\frac{1}{s} = \lim_{s \to 0} [1-G(s)] = 1 - G(0) \]

2) Closed Loop Case

\[ e_c(\infty) = \lim_{s \to 0} s \left[ \frac{1}{1+G(s)} \right] \frac{1}{s} = \frac{1}{1+G(0)} \]

\( G(0) \) is the dc gain and usually greater than 1. If \( G(0) \gg 1 \), closed-loop error \( e_c(\infty) \) is very small
<Example> Consider T.F.  \[ G(s) = \frac{K}{s+1} \]  Input \( R(s) = \frac{1}{s} \)

Open Loop case

\[ E_c(s) = [1 + G(s)]R(s) = [1 - G(s)]\frac{1}{s} \]

\[ e_0(\infty) = \lim_{s \to 0} E_c(s) = 1 - G(0) = 1 - K \]

Closed Loop Case

\[ E_c(s) = \frac{1}{1 + G(s)} R(s) = \frac{1}{1 + G(s)} \frac{1}{s} \]

\[ e_c(\infty) = \lim_{s \to 0} sE_c(s) = \frac{1}{1 + K} \]

1. For open loop if \( K = 1 \); \( e_0(\infty) = 0 \). However, during the operation the parameter of \( G(s) \) will change due to environment changes.

2. Closed loop error \( e_c(\infty) \) can be reduced by selecting high gain of \( K \) (if \( K = 100 \), \( e_c(\infty) = \frac{1}{101} \))

3. In case of the gain setting drifts or changes \( \frac{\Delta k}{k} = 0.1 \), open loop \( \Delta e_0(\infty) = 0.1 \) (10%) while closed loop: \( K = 100 \Rightarrow 90 \)

\[ \Delta e_c(\infty) = \frac{1}{101} - \frac{1}{91} = 0.0011 \quad (0.11\%) \]
SYSTEM ERROR

\[ T = \frac{G}{1 - (-GH)} = \frac{G}{1 + GH} \]

**Output** \[ Y(s) = T(s) \cdot R(s) \]

\[ Y(s) = \frac{G}{1 + GH} \cdot R(s) \]

Since \( Y(s) = E(s) \cdot G(s) \Rightarrow E(s) = \frac{Y(s)}{G(s)} \)

\[ E(s) = \frac{1}{1 + GH} \cdot R(s) \]

Actual error = Reference Input – Output \((R - Y)\)

**Case 1:** \( e_{ss} \) of unity feedback sys. \((H = 1)\)

\[ E = \frac{1}{1 + G} \cdot R = R - Y \]

It’s the actual error.

**Case 2:** \( e_{ss} \) of non-unity feedback sys. \((H \neq 1)\)

\[ E = \frac{1}{1 + G \cdot H} \cdot R = R - Y \cdot H \]

It’s not the actual error.

Use \( E = R(s) - Y(s) = R(s) - T(s) \cdot R(s) \)

\[ = [I - T(s)] \cdot R(s) \]
<Ex> find K for a zero steady-state error for $R(s) = \frac{1}{s}$

$T(s) = \frac{G}{1 + GH} = \frac{\frac{K}{(s + 2)}}{1 + \frac{\frac{2}{(s + 2)(s + 4)}}{(s + 2)}} = \frac{\frac{K(s + 4)}{s^2 + 6s + 8 + 2K}}{s + 4}$

Using $E(s) = (1 - T) R(s)$

$e_{ss} = \lim s (1 - T) \frac{1}{s} = 1 - T(0) = 1 - \frac{4k}{8 + 2k} = \frac{8 - 2k}{8 + 2k}$

For $e_{ss} = 0 \Rightarrow 8 - 2k = 0 \Rightarrow K = 4$