Duration, Convexity, and Immunization

\[ P(i) = \sum_{t=1}^{n} v^t R_t; \quad P'(i) = -\sum_{t=1}^{n} tv^{t+1} R_t; \quad P''(i) = \sum_{t=1}^{n} t(t+1)v^{t+2} R_t \]

Duration: \( \bar{d} = (\sum_{t=1}^{n} tv^t R_t)/(\sum_{t=1}^{n} v^t R_t) \)
Modified Duration: \( \bar{v} = -P'(i)/P(i) = \bar{d}/(1+i) \)
Convexity: \( \bar{c} = P''(i)/P(i) \)
We have \( \Delta \)-hedging
\[
P(i + h) \approx P(i)[1 - h\bar{v}] \]
\( \Gamma \)-hedging
\[
P(i + h) \approx P(i)[1 - h\bar{v} + \frac{h^2}{2}\bar{c}] \]

Example 1: A owes B $1100 at the end of one year and is required to set up an investment fund in order to meet this obligation. The only investments available are a money market fund earning 10% currently with the rate changing daily and two-year zero coupon bond also earning 10%. Assume the effective rate of interest is equal to 10% in all calculations.
(a) Develop an investment program based on immunization.
(b) Compute modified duration and convexity.
Let \( X \) be the amount invested in the money market fund and \( Y \) be the amount invested in two-year zero coupon bonds. Then we have
\[
P(i) = X + 1.21Y(1+i)^{-2} - 1100(1+i)^{-1} \]
\[
P'(i) = -2.42Y(1+i)^{-3} + 1100(1+i)^{-2} \]
\[
P''(i) = 7.26Y(1+i)^{-4} - 2200(1+i)^{-3} \]
For immunization (delta-hedging), we should have \( P(i) = P'(i) = 0 \), which leads to \( X = 500 \), and \( Y = 500 \). The modified duration and convexity are \( \bar{v} = 0.90909 \) and \( \bar{c} = 2.47934 > 0 \).

Example 2: The current price of an annual coupon bond is 100. The derivative of the price of the bond with respect to the yield to maturity is -700. The yield to maturity is an annual effective rate of 8%. Calculate the duration of the bond.
(A) 7.00 (B) 7.49 (C) 7.56 (D) 7.69 (E) 8.00
\[
\bar{d} = \bar{v} * (1+i) = (-P'(i)/P(i)) * (1+i) = 700 * 1.08/100 = 7.56
\]
Example 3: Calculate the duration of a common stock that pays dividends at the end of each year into perpetuity. Assume that the dividend increases by 2% each year and that the effective rate of interest is 5%.
(A) 27 (B) 35 (C) 44 (D) 52 (E) 58
\[
\bar{d} = \frac{\sum_{t=1}^{\infty} tv^t D(1.02)^{t-1}}{\sum_{t=1}^{\infty} v^t D(1.02)^{t-1}} = \frac{\sum_{t=1}^{\infty} tv^t(1.02)^t}{\sum_{t=1}^{\infty} v^t(1.02)^t} = \frac{\sum_{t=1}^{\infty} t(1.02/1.05)^t}{\sum_{t=1}^{\infty}(1.02/1.05)^t} = 35
\]