Solution for Homework 2

1. **Solution**: The LS estimate of μ is obtained by minimizing the following quantity

$$Q(\mu) = \sum_{i=1}^{n} (y_i - \mu)^2.$$

Notice that

$$\begin{aligned} \sum_{i=1}^{n} (y_i - \mu)^2 &= \sum_{i=1}^{n} [(y_i - \bar{y}) - (\bar{y} - \mu)]^2 \\ &= \sum_{i=1}^{n} [(y_i - \bar{y})^2 + (\bar{y} - \mu)^2 - 2 \cdot (y_i - \bar{y}) \cdot (\bar{y} - \mu)] \\ &= \sum_{i=1}^{n} (y_i - \bar{y})^2 + \sum_{i=1}^{n} (\bar{y} - \mu)^2 \end{aligned}$$

since the cross-product term is $\sum (y_i - \bar{y}) \cdot (\bar{y} - \mu) = 0$. It follows that

$$\sum_{i=1}^{n} (y_i - \mu)^2 \geq \sum_{i=1}^{n} (y_i - \bar{y})^2$$

with equality held if and only if $\mu = \bar{y}$. Namely, $Q(\mu)$ is minimized when $\mu = \bar{y}$.

2. **Solution**: First consider

$$\begin{aligned} \sum (x_i - \bar{x})(y_i - \bar{y}) &= \sum (x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\bar{y}) \\ &= \sum x_i y_i - \sum x_i \bar{y} - \sum \bar{x} y_i + \sum \bar{x}\bar{y} \\ &= \sum x_i y_i - n\bar{x}\bar{y} \end{aligned}$$

by noticing that $\sum x_i \bar{y} = \sum \bar{x} y_i = \sum \bar{x}\bar{y} = n\bar{x}\bar{y}$.
Now letting $y_i = x_i$ in the above results yields

$$\begin{aligned} LHS = \sum (x_i - \bar{x})(y_i - \bar{y}) &= \sum (x_i - \bar{x})(x_i - \bar{x}) \\ &= \sum (x_i - \bar{x})^2 \end{aligned}$$

$$\begin{aligned} RHS = \sum x_i y_i - n\bar{x}\bar{y} &= \sum x_i x_i - n\bar{x}\bar{x} \\ &= \sum x_i^2 - n \cdot \bar{x} \end{aligned}$$

i.e. $\sum (x_i - \bar{x})^2 = \sum x_i^2 - n \cdot \bar{x}^2$. Here, 'LHS' and 'RHS' are abbreviations for 'left hand side' and 'right hand side', respectively. Using similar arguments, we have

$$\sum (y_i - \bar{y})^2 = \sum y_i^2 - n \cdot \bar{y}^2$$

$$\sum(x_i-\bar{x})(y_i-\bar{y}) = \sum(x_i-\bar{x})\cdot y_i$$
$$\sum(x_i-\bar{x})^2 = \sum(x_i-\bar{x})x_i.$$

3. **Solution**: Note that $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \cdot \bar{x}$.

(a)

$$\begin{aligned}
\sum e_i &= \sum(y_i - \hat{y}_i) \\
&= \sum[y_i - (\hat{\beta}_0 + \hat{\beta}_1 \cdot x_i)] \\
&= \sum[y_i - (\bar{y} - \hat{\beta}_1 \cdot \bar{x} + \hat{\beta}_1 \cdot x_i)] \quad \text{bringing the expression for } \hat{\beta}_0 \\
&= \sum(y_i - \bar{y}) - \hat{\beta}_1 \sum(x_i - \bar{x}) \\
&= 0 - \hat{\beta}_1 \cdot 0 \\
&= 0
\end{aligned}$$

since $\sum(y_i - \bar{y}) = \sum(x_i - \bar{x}) = 0$.

(b) From identities above, $\sum e_i = \sum(y_i - \bar{y}) - \hat{\beta}_1 \sum(x_i - \bar{x})$. Hence,

$$\begin{aligned}
\sum e_i x_i &= \sum(y_i - \bar{y}) \cdot x_i - \hat{\beta}_1 \sum(x_i - \bar{x}) \cdot x_i \\
&= \sum(y_i - \bar{y}) \cdot (x_i - \bar{x}) - \hat{\beta}_1 \sum(x_i - \bar{x}) \cdot (x_i - \bar{x}) \\
&= 0
\end{aligned}$$

recalling that

$$\hat{\beta}_1 = \frac{\sum(y_i - \bar{y}) \cdot (x_i - \bar{x})}{\sum(x_i - \bar{x}) \cdot (x_i - \bar{x})}.$$

In the above inference, we also use the following results as stated in problem (2):

$$\sum(x_i-\bar{x})(y_i-\bar{y}) = \sum(x_i-\bar{x})y_i$$
$$\sum(x_i-\bar{x})^2 = \sum(x_i-\bar{x})x_i.¶$$