Homework 1

Exercise 1 An urn contains 6 red marbles and 4 black marbles. Two marbles are drawn *without replacement* from the urn. What is the probability that both of the marbles are black?

Solution: Let A = the event that the first marble is black; and let B = the event that the second marble is black. We know the following:

- In the beginning, there are 10 marbles in the urn, 4 of which are black. Therefore, P(A) = 4/10.
- After the first selection, there are 9 marbles in the urn, 3 of which are black. Therefore, P(B|A) = 3/9.

Therefore, based on the rule of multiplication:

$P(A \cap B) = P(A)\, P(B|A) = (4/10)*(3/9) = 12/90 = 2/15$

Exercise 2 Suppose we repeat the experiment of Exercise 1; but this time we select marbles *with replacement*. That is, we select one marble, note its color, and then replace it in the urn before making the second selection. When we select with replacement, what is the probability that both of the marbles are black?

Solution: Let A = the event that the first marble is black; and let B = the event that the second marble is black. We know the following:

- In the beginning, there are 10 marbles in the urn, 4 of which are black. Therefore, P(A) = 4/10.
- After the first selection, we replace the selected marble; so there are still 10 marbles in the urn, 4 of which are black. Therefore, P(B|A) = 4/10.

Therefore, based on the rule of multiplication:

$P(A \cap B) = P(A)\, P(B|A)$

$P(A \cap B) = (4/10)*(4/10) = 16/100 = 4/25$

Example 3 Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on the day of Marie's wedding?

Solution: The sample space is defined by two mutually-exclusive events - it rains or it does not rain. Additionally, a third event occurs when the weatherman predicts rain. Notation for these events appears below.

- Event A_1. It rains on Marie's wedding.
- Event A_2. It does not rain on Marie's wedding
- Event B. The weatherman predicts rain.

In terms of probabilities, we know the following:

- $P(A_1) = 5/365 = 0.0136985$ [It rains 5 days out of the year.]
- $P(A_2) = 360/365 = 0.9863014$ [It does not rain 360 days out of the year.]
- $P(B \mid A_1) = 0.9$ [When it rains, the weatherman predicts rain 90% of the time.]
- $P(B \mid A_2) = 0.1$ [When it does not rain, the weatherman predicts rain 10% of the time.]

We want to know $P(A_1 \mid B)$, the probability it will rain on the day of Marie's wedding, given a forecast for rain by the weatherman. The answer can be determined from Bayes' theorem, as shown below.

$$P(A_1 \mid B) = \frac{P(A_1)\,P(B \mid A_1)}{P(A_1)\,P(B \mid A_1) + P(A_2)\,P(B \mid A_2)}$$
$$= (0.014)(0.9) / [\,(0.014)(0.9) + (0.986)(0.1)\,]$$
$$= 0.111$$

Note the somewhat unintuitive result. Even when the weatherman predicts rain, it only rains only about 11% of the time. Despite the weatherman's gloomy prediction, there is a good chance that Marie will not get rained on at her wedding.

Example 4 An average light bulb manufactured by the Acme Corporation lasts 300 days with a standard deviation of 50 days. Assume that bulb life, denoted by X, is normally distributed.

a) If we express the life time of a light bulb in terms of years (denote it as X') what would be its distribution?

Solution: First of all, we know that $X \sim N(300, 50^2)$. When expressed in years, $X' = X/365 = 0 + \frac{1}{365}X$, which is a linear combination of X and hence also follows the normal distribution.

Furthermore, we have

$$\begin{cases} E(X') = 0 + \frac{1}{365}E(X) = 300/365 = 0.8219 \\ Var(X') = Var(X)/365^2 = 50^2/365^2 = 0.018765 \end{cases}$$

Thus $X' \sim N(0.8219,\ 0.018765)$.

b) What is the probability that an Acme light bulb will last at most 365 days?

Solution: Given a mean score of 300 days and a standard deviation of 50 days, we want to find the cumulative probability that bulb life is less than or equal to 365 days. Thus, we know the following:

- The value of the normal random variable is 365 days.
- The mean is equal to 300 days.
- The standard deviation is equal to 50 days.

We enter these values into the Normal Distribution Calculator and compute the cumulative probability. The answer is: $P(X \leq 365) = 0.90$. Hence, there is a 90% chance that a light bulb will burn out within 365 days.

Exercise 5 Suppose scores on an IQ test are normally distributed. If the test has a mean of 100 and a standard deviation of 10, what is the probability that a person who takes the test will score between 90 and 110?

Solution: Here, we want to know the probability that the test score falls between 90 and 110. The "trick" to solving this problem is to realize the following:

$$P(90 < X < 110) = P(X < 110) - P(X < 90)$$

We use the Normal Distribution Calculator to compute both probabilities on the right side of the above equation.

- To compute $P(X < 110)$, we enter the following inputs into the calculator: The value of the normal random variable is 110, the mean is 100, and the standard deviation is 10. We find that $P(X < 110)$ is 0.84.
- To compute $P(X < 90)$, we enter the following inputs into the calculator: The value of the normal random variable is 90, the mean is 100, and the standard deviation is 10. We find that $P(X < 90)$ is 0.16.

We use these findings to compute our final answer as follows:

$$P(90 < X < 110) = P(X < 110) - P(X < 90) = 0.84 - 0.16 = 0.68$$

Thus, about 68% of the test scores will fall between 90 and 110.

Exercise 6: In hypothesis testing, which of the following statements are always true?

I. The P-value is greater than the significance level.
II. The P-value is computed from the significance level.
III. The P-value is the parameter in the null hypothesis.
IV. The P-value is a test statistic.
V. The P-value is a probability.

(A) I only
(B) II only
(C) III only
(D) IV only
(E) V only

Solution The correct answer is (E). The P-value is the probability of observing a sample statistic as extreme as the test statistic. It can be greater than the significance level, but it can also be smaller than the significance level. It is not computed from the significance level, it is not the parameter in the null hypothesis, and it is not a test statistic.

Exercise 7: (A *Two-Tailed t Test Example*) An inventor has developed a new, energy-efficient lawn mower engine. He claims that the engine will run continuously for 5 hours (300 minutes) on a single gallon of regular gasoline. Suppose a simple random sample of 50 engines is tested. The engines run for an average of 295 minutes, with a standard deviation of 20 minutes. Test the null hypothesis that the mean run time is 300 minutes against the alternative hypothesis that the mean run time is not 300 minutes. Use a 0.05 level of significance. (Assume that run times for the population of engines are normally distributed.)

Solution: The solution to this problem takes four steps: (1) state the hypotheses, (2) formulate an analysis plan, (3) analyze sample data, and (4) interpret results. We work through those steps below:

- **State the hypotheses.** The first step is to state the null hypothesis and an alternative hypothesis.
  - Null hypothesis: $\mu = 300$
  - Alternative hypothesis: $\mu \neq 300$

  Note that these hypotheses constitute a two-tailed test. The null hypothesis will be rejected if the sample mean is too big or if it is too small.

- **Formulate an analysis plan**. For this analysis, the significance level is 0.05. The test method is a one-sample t-test.

- **Analyze sample data**. Using sample data, we compute the standard error (SE), degrees of freedom (DF), and the t-score test statistic (t).

  SE = s / sqrt(n) = 20 / sqrt(50) = 20/7.07 = 2.83
  DF = n - 1 = 50 - 1 = 49
  t = (x - μ) / SE = (295 - 300)/2.83 = 1.77

  where s is the standard deviation of the sample, x is the sample mean, μ is the hypothesized population mean, and n is the sample size.

  Since we have a two-tailed test, the P-value is the probability that the t-score having 49 degrees of freedom is less than -1.77 or greater than 1.77.

  We use the t Distribution table to find P(t < -1.77) = 0.04, and P(t > 1.75) = 0.04. Thus, the P-value = 0.04 + 0.04 = 0.08.

- **Interpret results**. Since the P-value (0.08) is greater than the significance level (0.05), we cannot reject the null hypothesis.

Problem 8: (*A One-Tailed t Test Example*) Bon Air Elementary School has 300 students. The principal of the school thinks that the average IQ of students at Bon Air is at least 110. To prove her point, she administers an IQ test to 20 randomly selected students. Among the sampled students, the average IQ is 108 with a standard deviation of 10. Based on these results, should the principal accept or reject her original hypothesis? Assume a significance level of 0.01.

Solution: The solution to this problem takes four steps: (1) state the hypotheses, (2) formulate an analysis plan, (3) analyze sample data, and (4) interpret results. We work through those steps below:

- **State the hypotheses.** The first step is to state the null hypothesis and an alternative hypothesis.

  Null hypothesis: $\mu >= 110$

  Alternative hypotheses: $\mu < 110$

  Note that these hypotheses constitute a one-tailed test. The null hypothesis will be rejected if the sample mean is too small.

- **Formulate an analysis plan**. For this analysis, the significance level is 0.01. The test method is a one-sample t-test.

- **Analyze sample data**. Using sample data, we compute the standard error (SE), degrees of freedom (DF), and the t-score test statistic (t).

  SE = s / sqrt(n) = 10 / sqrt(20) = 10/4.472 = 2.236

  DF = n - 1 = 20 - 1 = 19

  $t = (x - \mu)$ / SE = (108 - 110)/2.236 = -0.894

  where s is the standard deviation of the sample, x is the sample mean, μ is the hypothesized population mean, and n is the sample size.

  Since we have a one-tailed test, the P-value is the probability that the t-score having 19 degrees of freedom is less than -0.894.

  We use the t Distribution to find $P(t < -0.894) = 0.19$. Thus, the P-value is 0.19.

- **Interpret results**. Since the P-value (= 0.19) is greater than the significance level (0.01), we cannot reject the null hypothesis.