Step-down FDR Procedures for Large Numbers of Hypotheses

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Abstract. Somerville (2004b) developed FDR step-down procedures which were particularly appropriate for cases where the number of false hypotheses was small. The test statistics were assumed to have a multivariate-t distribution with common correlation. MCV's (minimum critical values) were chosen so that 8 unique critical values resulted. Tables were given for numbers of hypotheses m, ranging from 50 to 10,000, for $\rho = 0., 0.5$, and $\nu = 15, \infty$. In this paper we extend the results, using MCV's resulting in 31 critical values. Tables are given for the same values of m, for $\rho = 0, 0.1, 0.5$ and $\nu = 15, \infty$. Interpolation rules are given for m, ρ and ν. Use of larger numbers of critical values increase both the power and the number of hypotheses falsely rejected. When the expected number of false hypotheses is small, use of the procedures of this paper results in a reduced number of false rejections with a negligible reduction in power.

1 Introduction

There are many situations where a researcher is interested in the outcome of a family of tests. An example is the case where m experimental drugs are compared to a standard with respect to an outcome. Tests can be made of the m null hypotheses of no effect, each at a level α. This would be testing at a per comparison error rate (PCER) of α. For $m > 1$, the probability of rejecting some null hypothesis when it was true would be larger than α. To protect from such situations arising from multiplicity, a standard procedure has been to use a family-wise error rate (FWER). In this case, the family-wise error rate would be the probability of rejecting at least one null hypothesis when in fact they are all true. A common single-step method to achieve the family-wise error rate is to use the Bonferroni procedure, discussed by R. A. Fisher (1935). Other single step procedures include those of Scheffe (1953) for testing several linear combinations, Tukey (1953) for pair-wise comparisons and Dunnett (1955) for comparisons with a control.

Multi-step (step-wise) FWER procedures have been developed which are more powerful than single-step procedures. One of the first was introduced by Naik (1975). Others include step-up and step-down procedures developed for comparisons with a control by Dunnett and Tamhane (1991, 1992).

More recently multi-step procedures which control the “false discovery rate" FDR (expected proportion of “false discoveries" or Type I errors) say q have been proposed. One motivation was the under-utilization of FWER procedures because of their perceived lack of “power". Benjamini and Hochberg (1995) presented a step-up procedure, valid for independent test statistics. Benjamini and Liu (1999a) presented a step-down procedure valid under the same conditions. Troendle (2000) developed both step-up and step-down procedures which asymptotically control the FDR when the test statistics have a multivariate-t distribution. Benjamini and Liu (1999b) and Benjamini and Yekutieli (1999) gave distribution free FDR procedures. Benjamini and Yekutieli (2001) showed that the procedure of Benjamini and Hochberg (1995) was also valid when the p-values were “positively dependent". Benjamini, Krieger and Yekutieli (2001) developed a two stage step-up procedure. Sarkar (2002) has made important contributions.
Somerville (2004b), using least favorable configurations, developed both step-up and step-down procedures which are valid for dependent or independent hypotheses. All calculations assume the test statistics have a joint multivariate-t distribution and a common correlation coefficient ρ. The concept of “Minimum Critical Value" (MCV) was introduced. When the MCV (see section 2.) is equal to 0 the step-down procedure is the same as that of Troendle (2000). Studies comparing powers of various procedures by Horne and Dunnett (2003) and Somerville (2003) have shown the methods of Troendle (2000) and Somerville (2004b) to be among the most powerful.
FDR procedures, while controlling the expected number of “false positives", have also been criticized for not controlling the actual number or proportion. van der Laan, Dudoit and Pollard (2004) generalize FWER procedures, introducing gFWER which controls u, the number of Type I errors. They also introduce PFP to control γ, the proportion of “false positives".
In this paper we introduce step-down FDR procedures which use large MCV’s (Minimum Critical Values). These procedures not only control the expected FDR, but also result in values of $P[u \leq 2]$ which compare with those achieved by Korn et al and van der Laan et al when the number of false hypotheses is small, or where finding a small number of false hypotheses is a satisfactory outcome. Six tables of critical values d_{m-30} to d_m are given for $50 \leq m \leq 10,000$. Instead of using $MCV = d_{m-30}$, and 31 “unique" critical values, the tables may be “truncated" by elimination of the smallest values. Reducing the number of “unique" critical values, reduces the probability of “a or fewer" false positives, where the value of a is arbitrary.

2 Minimum Critical Values

Assuming a multivariate-t distribution of the test statistics with a common correlation coefficient ρ, Somerville (2003) observed that, in most situations, the smallest of the calculated FDR critical values was less than the commonly used critical value of $t_{\alpha,\nu}$ for comparing two means, where $t_{\alpha,\nu}$ is the value of student’s t with type I error of α and ν degrees of freedom. In many situations it was negative, and when the number of tests was large, one or more of the critical values could be negatively infinite. This situation would certainly be unappealing to most users, and motivated the concept of “minimum critical value" (MCV). The “minimum critical value" is defined to be the

smallest critical value, which, when exceeded or equaled, would result in an hypothesis rejection. The “minimum critical value" may be more or less arbitrarily chosen. If the calculated value for d_i is smaller than the chosen MCV, it is replaced by the MCV value, and d_{i+1}, d_{i+2}, are sequentially calculated, where $d_i \leq d_j$ when $i \leq j$.
Somerville (2003, 2004a,b) made many calculations finding convincing empirical evidence, especially for large values of m, that if n_F were the actual number of false hypotheses, that there was an inverse relationship between n_F and the value of the MCV which maximized the “power" of the FDR procedure. That is, if n_F were large, the MCV for maximum power of the procedure should be small and for small values of n_F , the MCV should be large. In particular, Somerville observed that, for the values of m studied, choosing the value of MCV such that the number of “unique" critical values was equal to n_F resulted in “powers" near the maximum. An approximately equivalent result could be obtained by automatically “truncating" the step-down FDR process after n_F steps. It may be noted (Somerville (2003, 2004b)) that if the MCV is equal to the corresponding critical value for the single-step test, all the critical values of the FDR procedure are equal.

3 Tables for the FDR Step-down Procedure when the number of False Hypotheses is Small

Tables 2, 3 and 4 give step-down FDR critical values for m ranging from 50 to 10,000 when $\nu = 15$ or $\infty, q = .05$ for the procedure, and ρ = 0, .1 and .5 respectively, when the hypotheses are one-sided. For each combination of m, ρ and ν. MCV was chosen as the smallest value such that $d_1 = d_2 = \ldots = d_{m-30} = MCV$ results in the FDR less than or equal to q. There are thus, for each m in each table, exactly 31 “unique" critical values, a more or less arbitrarily chosen “small" value.
There may be error in the third decimal place in the tables. Critical values for m not included in the tables can be obtained by interpolation (or extrapolation if $m < 50,000$ (say)). Each critical value d_i in a table is approximately linear in $ln(m)$. Critical values for $15 < \nu < \infty$ can be obtained by linear interpolation in 1/ν. Critical values for $0 < \rho < .5$, can be approximated using quadratic interpolation. It is worth noting that as $\rho \rightarrow 1$, all critical values are equal to z_q (or t_q), the value of the normal (or t) variate which is exceeded q of the time.

4 Calculation of critical values and powers

Fortran 90 programs SEQDN and SEQUP can sequentially calculate the critical values d_2 to d_m for step-down and step-up FDR, respectively, for arbitrary values of m, q, ρ, and ν. N ($10^5, 10^6$ or 10^7) random normal multivariate vectors of size m were used to obtain each critical value.
Fortran 90 programs FDRPWRDN and FDRPWRUP calculate powers, E(Q), $P[u \leq 1, 2, \ldots, 7]$, and $P[\gamma \leq .05, .10, .15]$ where u and γ are the number and proportion of false discoveries respectively. Inputs are $m, \rho, \nu, \Delta, n_F$ and a set of m critical values. Δ is the common standardized mean of the test statistics corresponding to the n_F false

hypotheses. N random normal multivariate vectors of size m are used. Three kinds of power are always calculated: per pair, all pairs and any pair (see Horn and Dunnett (2004)), and also E(Q). The probability of rejecting at least one of the false hypotheses is called any pairs power. The probability of rejecting all false hypotheses is called all pairs power. Considering a specific hypothesis, the probability of its rejection is called the per pairs power. Since our calculations assume all the test statistics corresponding to the F hypotheses have the same location parameter, the per pair power is identical to the average power.

5 Example

Korn, Troendle, McShane and Simon (2003) recently proposed two new procedures which control, with specified confidence, the actual number of false discoveries, and the actual proportion of false discoveries, respectively. They applied their procedures to analyze a microarray dataset consisting of measurements on approximately 9000 genes in paired tumor specimens, collected both before and after chemotherapy on 20 breast cancer patients. Their study, after elimination of cases of missing data included 8029 genes for analysis. The object was to simultaneously test the null hypotheses that the mean pre and post chemotherapy expression of genes was the same. Their Procedure A identified 28 genes where u, the number of false discoveries, was ≤ 2, with confidence .05. Procedure B identified the same 28 genes where γ, the false discovery proportion, was $\leq$ .10 with approximate confidence .95.

Table 1. Table A: Number of genes identified by procedures, $(m = 8029, q = 0.05)$. Table B: Minimum values of $P[u \leq 2]$ $(m = 8029, q = .05, \nu = 3.464)$. Rough approximate value of n_F at minimum in parentheses.

Table A

	$\nu = \infty$		$\nu = 15$	
MCV	$\rho = .0$	$\rho = .1$	$\rho = .5$	$\rho = .0$
d_{m-7}	20	21	29	24
d_{m-11}	23	23	33	27
d_{m-15}	24	24	35	29
d_{m-19}	27	27	38	31
d_{m-23}	28	29	40	33
d_{m-27}	29	29	> 50	33
d_{m-30}	29	33	> 50	36

Table B

	$\nu = \infty$		$\nu = 15$	
MCV	$\rho = .0$	$\rho = .1$	$\rho = .5$	$\rho = .0$
d_{m-7}	.99(40)	.96(70)	.95(50)	.90(50)
d_{m-11}	.98(50)	.94(70)	.93(55)	.90(50)
d_{m-15}	.96(70)	.91(70)	.92(55)	.89(80)
d_{m-19}	.92(80)	.88(70)	.91(60)	.89(80)
d_{m-23}	.88(80)	.85(70)	.91(60)	.88(90)
d_{m-27}	.83(80)	.82(80)	.90(60)	.88(90)
d_{m-30}	.78(90)	.78(90)	.87(70)	.88(100)

The tables 2, 3 and 4 were used to test the same hypotheses, “truncating" each table to utilize exactly 8, 12, 16, 20, 24, 28 and 31 “unique" critical values (“truncating" to exactly 12 “unique" critical values would mean setting MCV equal to d_{m-11}, and setting $d_i = MCV$ for $i < m - 11$.). Assuming 15 degrees of freedom for each of the 8029 tests, the corresponding FDR step-down critical values were obtained using

Table 2. Step-down FDR Critical Values (31 “unique") for $\rho = 0$ and $q = 0.05$ (the first columns should be read as d_i where $i = m, m-1, \ldots, m-30, \ldots, 1$)

Table C $(\rho = 0, \nu = 15, q = 0.05)$

m	50	100	250	500	1000	2500	5000	10000
m	3.632	3.928	4.306	4.579	4.842	5.177	5.420	5.656
$m-1$	3.332	3.647	4.045	4.332	4.614	4.962	5.214	5.464
$m-2$	3.140	3.467	3.876	4.173	4.458	4.822	5.081	5.340
$m-3$	2.999	3.335	3.753	4.054	4.345	4.710	4.976	5.225
$m-4$	2.885	3.231	3.656	3.961	4.254	4.630	4.899	5.150
$m-5$	2.789	3.141	3.573	3.884	4.182	4.560	4.827	5.100
$m-6$	2.705	3.065	3.503	3.816	4.066	4.444	4.723	4.979
$m-7$	2.630	2.999	3.443	3.760	4.066	4.444	4.723	4.979
$m-8$	2.561	2.937	3.389	3.705	4.013	4.392	4.672	4.941
$m-9$	2.499	2.885	3.338	3.660	3.967	4.357	4.631	4.912
$m-10$	2.441	2.835	3.293	3.617	3.929	4.306	4.585	4.847
$m-11$	2.383	2.787	3.249	3.577	3.886	4.283	4.567	4.834
$m-12$	2.333	2.743	3.213	3.543	3.855	4.244	4.527	4.803
$m-13$	2.282	2.702	3.176	3.504	3.824	4.220	4.488	4.774
$m-14$	2.233	2.666	3.144	3.474	3.784	4.182	4.459	4.746
$m-15$	2.187	2.627	3.111	3.446	3.759	3.153	4.441	4.735
$m-16$	2.141	2.591	3.081	3.416	3.731	4.124	4.426	4.677
$m-17$	2.079	2.559	3.051	3.388	3.704	4.113	4.394	4.677
$m-18$	2.052	2.526	3.025	3.362	3.679	4.078	4.359	4.645
$m-19$	2.010	2.495	3.000	3.338	3.669	4.063	4.346	4.625
$m-20$	1.969	2.464	3.973	3.315	3.636	4.025	4.329	4.602
$m-21$	1.926	2.436	2.949	3.292	3.605	4.024	4.303	4.577
$m-22$	1.884	2.406	2.926	3.272	3.592	3.994	4.279	4.567
$m-23$	1.842	2.380	2.905	3.250	3.570	3.980	4.249	4.532
$m-24$	1.801	2.352	2.883	3.227	3.554	3.952	4.238	4.532
$m-25$	1.759	2.328	2.861	3.207	3.538	3.937	4.221	4.507
$m-26$	1.716	2.302	2.839	3.192	3.513	3.925	4.203	4.494
$m-27$	1.670	2.275	2.822	3.169	3.493	3.890	4.185	4.457
$m-28$	1.625	2.249	2.797	3.152	3.465	3.875	4.185	4.445
$m-29$	1.625	2.243	2.790	3.138	3.454	3.842	4.136	4.408
$m-30$	1.464	2.152	2.712	3.062	3.385	3.784	4.065	4.343
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1	1.464	2.152	2.712	3.062	3.385	3.784	4.065	4.343
$ln(m)$	3.912	4.605	5.521	6.215	6.908	7.824	8.517	9.210

Table D $(\rho = 0, \nu = \infty, q = 0.05)$

m	50	100	250	500	1000	2500	5000	10000
m	3.083	3.283	3.533	3.713	3.884	4.102	4.259	4.412
$m-1$	2.867	3.081	3.346	3.537	3.714	3.938	4.103	4.266
$m-2$	2.731	2.958	3.234	3.426	3.613	3.842	4.009	4.172
$m-3$	2.629	2.865	3.150	3.349	3.538	3.772	3.940	4.106
$m-4$	2.545	2.791	3.084	3.286	3.477	3.713	3.887	4.054
$m-5$	2.474	2.729	3.027	3.234	3.428	3.672	3.842	4.011
$m-6$	2.412	2.676	2.980	3.190	3.387	3.635	3.804	3.974
$m-7$	2.354	2.626	2.938	3.150	3.350	3.596	3.771	3.941
$m-8$	2.302	2.585	2.899	3.116	3.317	3.566	3.742	3.915
$m-9$	2.254	2.543	2.865	3.084	3.286	3.543	3.717	3.887
$m-10$	2.207	2.507	2.835	3.054	3.260	3.511	3.694	3.865
$m-11$	2.165	2.470	2.806	3.029	3.235	3.488	3.669	3.842
$m-12$	2.122	2.440	2.777	3.003	3.213	3.468	3.655	3.824
$m-13$	2.082	2.407	2.753	2.980	3.190	3.448	3.633	3.803
$m-14$	2.044	2.379	2.730	2.960	3.169	3.428	3.613	3.788
$m-15$	2.006	2.352	2.705	2.937	3.151	3.411	3.602	3.773
$m-16$	1.969	2.324	2.684	2.919	3.133	3.399	3.582	3.757
$m-17$	1.932	2.300	2.666	2.901	3.117	3.376	3.566	3.743
$m-18$	1.897	2.274	2.644	2.881	3.099	3.365	3.557	3.728
$m-19$	1.861	2.251	2.625	2.867	3.084	3.354	3.537	3.719
$m-20$	1.826	2.223	2.609	2.849	3.069	3.334	3.526	3.704
$m-21$	1.791	2.206	2.590	2.833	3.055	3.323	3.516	3.693
$m-22$	1.755	2.181	2.573	2.821	3.041	3.308	3.502	3.685
$m-23$	1.720	2.160	2.557	2.804	3.029	3.308	3.484	3.672
$m-24$	1.685	2.138	2.541	2.792	3.016	3.282	3.484	3.660
$m-25$	1.648	2.121	2.531	2.780	3.003	3.275	3.469	3.654
$m-26$	1.621	2.098	2.509	2.764	2.994	3.269	3.455	3.644
$m-27$	1.549	2.066	2.490	2.746	2.976	3.249	3.443	3.628
$m-28$	1.549	2.066	2.490	2.746	2.969	3.249	3.443	3.628
$m-29$	1.549	2.066	2.490	2.746	2.969	3.249	3.443	3.628
$m-30$	1.397	1.991	2.437	2.700	2.935	3.212	3.408	3.594
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1	1.397	1.991	2.437	2.700	2.935	3.212	3.408	3.594
$ln(m)$	3.912	4.605	5.521	6.215	6.908	7.824	8.517	9.210

the Fortran program SEQDN. Table 1A indicates the number of genes identified by the procedures.

Table 1B illustrates the control of the number of false positives, respectively for $m = 8029$. The graph $P[u \leq 2]$ vs. n_F is (shallow) bowl shaped. The value in parentheses is a very rough estimate of the value of n_F for the minimum.

6 Summary and Conclusion

Step-down FDR procedures have been developed for the case where there are many hypotheses. The procedures are particularly appropriate when relatively few hypotheses are false, or where obtaining a limited number of “discoveries" is satisfactory. For $50 \leq m \leq 10,000$ and q = .05, tables of critical values of the test statistics are presented for $\rho = 0$, .1 and .5 when $\nu = \infty$, or $\nu = 15$. There are 31 “unique" tabulated critical values, but the tables may be “truncated". Setting MCV equal to d_m results in a single unique critical value and a single step procedure. This choice yields the largest $P[u \leq a]$ where a is arbitrary. Choosing the appropriate value for MCV is an attempt to balance the conflicting goals of rejecting all false hypotheses and not rejecting those hypotheses which are true. Empirical results suggest that “powers" are maximized when the MCV

Table 3. Step-down FDR Critical Values (31 “unique") for $\rho = 0.1$ and $q = 0.05$ (the first columns should be read as d_i where $i = m, m-1, \ldots, m-30, \ldots, 1$)

Table E $(\rho = 0.1, \nu = 15, q = 0.05)$

m	50	100	250	500	1000	2500	5000	10000
m	3.580	3.860	4.217	4.474	4.719	5.033	5.260	5.479
$m-1$	3.295	3.597	3.975	4.245	4.505	4.831	5.074	5.296
$m-2$	3.113	3.427	3.816	4.097	4.364	4.703	4.947	5.171
$m-3$	2.987	3.300	3.701	3.985	4.261	4.597	4.848	5.092
$m-4$	2.867	3.201	3.608	3.899	4.176	4.536	4.777	5.018
$m-5$	2.774	3.115	3.530	3.825	4.107	4.459	4.711	4.957
$m-6$	2.693	3.041	3.465	3.764	4.050	4.406	4.662	4.900
$m-7$	2.619	2.980	3.405	3.706	3.997	4.355	4.597	4.853
$m-8$	2.552	2.918	3.356	3.658	3.934	4.302	4.581	4.822
$m-9$	2.491	2.868	3.306	3.612	3.906	4.276	4.522	4.778
$m-10$	2.434	2.820	3.264	3.572	3.851	4.223	4.507	4.742
$m-11$	2.379	2.773	3.220	3.535	3.834	4.202	4.451	4.722
$m-12$	2.328	2.730	3.186	3.502	3.796	4.159	4.428	4.691
$m-13$	2.279	2.691	3.150	3.466	3.759	4.144	4.395	4.650
$m-14$	2.231	2.655	3.119	3.434	3.742	4.102	4.395	4.623
$m-15$	2.186	2.617	3.086	3.406	3.708	4.066	4.350	4.623
$m-16$	2.141	2.583	3.058	3.380	3.679	4.054	4.333	4.573
$m-17$	2.098	2.549	3.028	3.350	3.648	4.054	4.299	4.556
$m-18$	2.054	2.519	3.003	3.330	3.641	4.001	4.281	4.545
$m-19$	2.010	2.486	2.980	3.307	3.601	3.986	4.253	4.545
$m-20$	1.973	2.456	2.952	3.278	3.601	3.958	4.237	4.495
$m-21$	1.929	2.432	2.929	3.260	3.563	3.958	4.213	4.484
$m-22$	1.888	2.400	2.908	3.238	3.538	3.919	4.213	4.440
$m-23$	1.847	2.375	2.884	3.218	3.527	3.913	4.188	4.439
$m-24$	1.807	2.346	2.863	3.195	3.501	3.889	4.162	4.419
$m-25$	1.765	2.324	2.841	3.176	3.494	3.865	4.136	4.419
$m-26$	1.720	2.297	2.820	3.156	3.457	3.853	4.124	4.396
$m-27$	1.682	2.269	2.801	3.135	3.439	3.833	4.101	4.355
$m-28$	1.634	2.244	2.776	3.116	3.425	3.806	4.094	4.350
$m-29$	1.622	2.229	2.760	3.099	3.425	3.780	4.046	4.307
$m-30$	1.453	2.133	2.681	3.020	3.325	3.708	3.978	4.240
.	.	.	.	.	.	.	.	
1	1.453	2.133	2.681	3.020	3.325	3.708	3.978	4.240
$ln(m)$	3.912	4.605	5.521	6.215	6.908	7.824	8.517	9.210

Table F $(\rho = 0.1, \nu = \infty, q = 0.05)$

m	50	100	250	500	1000	2500	5000	10000
m	3.070	3.268	3.516	3.693	3.864	4.081	4.235	4.386
$m-1$	2.858	3.072	3.335	3.522	3.700	3.924	4.086	4.242
$m-2$	2.724	2.949	3.224	3.416	3.599	3.829	3.995	4.153
$m-3$	2.623	2.859	3.141	3.339	3.526	3.759	3.929	4.091
$m-4$	2.542	2.786	3.076	3.277	3.468	3.704	3.880	4.041
$m-5$	2.471	2.725	3.021	3.226	3.420	3.660	3.832	3.999
$m-6$	2.409	2.670	2.974	3.181	3.377	3.621	3.795	3.963
$m-7$	2.352	2.623	2.932	3.142	3.343	3.588	3.763	3.932
$m-8$	2.300	2.580	2.895	3.107	3.309	3.558	3.735	3.907
$m-9$	2.253	2.541	2.862	3.077	3.280	3.531	3.708	3.876
$m-10$	2.207	2.503	2.829	3.049	3.255	3.503	3.685	3.858
$m-11$	2.164	2.469	2.801	3.023	3.227	3.481	3.662	3.834
$m-12$	2.122	2.437	2.774	2.998	3.205	3.461	3.643	3.817
$m-13$	2.084	2.407	2.749	2.975	3.185	3.442	3.625	3.797
$m-14$	2.045	2.377	2.725	2.954	3.161	3.424	3.609	3.778
$m-15$	2.008	2.350	2.703	2.934	3.147	3.403	3.586	3.766
$m-16$	1.970	2.324	2.681	2.914	3.127	3.389	3.572	3.749
$m-17$	1.937	2.298	2.661	2.896	3.110	3.372	3.560	3.735
$m-18$	1.899	2.274	2.641	2.878	3.096	3.359	3.542	3.720
$m-19$	1.866	2.250	2.623	2.862	3.078	3.343	3.533	3.710
$m-20$	1.830	2.226	2.605	2.845	3.064	3.331	3.513	3.698
$m-21$	1.795	2.204	2.587	2.831	3.052	3.313	3.506	3.680
$m-22$	1.760	2.182	2.572	2.813	3.033	3.309	3.491	3.672
$m-23$	1.727	2.161	2.555	2.803	3.025	3.289	3.488	3.663
$m-24$	1.691	2.138	2.539	2.786	3.008	3.278	3.464	3.652
$m-25$	1.654	2.121	2.523	2.773	2.997	3.269	3.459	3.639
$m-26$	1.622	2.099	2.510	2.760	2.985	3.259	3.449	3.630
$m-27$	1.574	2.074	2.492	2.747	2.969	3.243	3.439	3.623
$m-28$	1.544	2.059	2.480	2.734	2.969	3.239	3.426	3.607
$m-29$	1.544	2.059	2.480	2.733	2.956	3.233	3.424	3.593
$m-30$	1.388	1.978	2.420	2.681	2.912	3.187	3.381	3.568
.	.	.	.	.	.	.	.	.
1	1.388	1.978	2.420	2.681	2.912	3.187	3.381	3.568
$ln(m)$	3.912	4.605	5.521	6.215	6.908	7.824	8.517	9.210

is chosen so that the number of “unique" critical values is equal to n_F (of course almost always unknown).
Some additional observations and conclusions are:

1. i) $P[u \leq a]$ decreases as the number of unique critical values used increases. Thus the number of critical values can be chosen so as to control $P[u \leq a]$.
2. ii) The number of hypotheses rejected increases with the number of “unique" critical values used in the step-down procedure. The probability of false rejection also increases.
3. iii) For sufficiently large n_F, as n_F increases $P[u \leq 2]$ increases and approaches 1.
4. iv) The tables have been calculated under the assumption that the test statistics have a joint multivariate-t distribution with common correlation coefficient ρ. However, the critical values tabulated for the test statistic can be converted to critical p-values. These converted p-values may be useful for cases where the joint distribution of the test statistics is other than multivariate-t or that any individual test is one- two sided.
5. v) Use of the step-down FDR procedure of this paper is an alternative to procedure A of Korn et al and the generalized procedure of van der Laan, Dudoit and Pollard.
6. vi) Underestimating ρ results in the identification of fewer false hypotheses.

Table 4. Step-down FDR Critical Values (31 “unique") for $\rho = 0.5$ and $q = 0.05$ (the first columns should be read as d_i where $i = m, m-1, \ldots, m-30, \ldots, 1$)

Table G$(\rho = 0.5, \nu = 15, q = 0.05)$

m	50	100	250	500	1000	2500	5000	10000
m	3.243	3.452	3.711	3.896	4.067	4.294	4.455	4.608
$m-1$	3.039	3.266	3.545	3.738	3.929	4.155	4.322	4.483
$m-2$	2.902	3.144	3.432	3.634	3.826	4.057	4.242	4.409
$m-3$	2.798	3.048	3.349	3.558	3.750	3.990	4.164	4.328
$m-4$	2.713	2.974	3.284	3.497	3.693	3.943	4.118	4.303
$m-5$	2.641	2.909	3.223	3.444	3.648	3.905	4.076	4.255
$m-6$	2.575	2.853	3.179	3.397	3.607	3.865	4.038	4.191
$m-7$	2.514	2.805	3.129	3.358	3.563	3.812	4.004	4.161
$m-8$	2.460	2.755	3.096	3.314	3.528	3.786	3.972	4.133
$m-9$	2.410	2.717	3.056	3.289	3.494	3.758	3.944	4.106
$m-10$	2.362	2.678	3.026	3.256	3.468	3.732	3.918	4.080
$m-11$	2.316	2.642	2.989	3.233	3.444	3.709	3.894	4.056
$m-12$	2.274	2.606	2.963	3.199	3.420	3.690	3.873	4.033
$m-13$	2.232	2.571	2.937	3.181	3.397	3.670	3.853	4.011
$m-14$	2.189	2.543	2.913	3.145	3.375	3.642	3.835	3.991
$m-15$	2.152	2.514	2.878	3.131	3.354	3.623	3.819	3.972
$m-16$	2.111	2.478	2.864	3.103	3.333	3.605	3.803	3.954
$m-17$	2.077	2.452	2.836	3.095	3.313	3.588	3.789	3.938
$m-18$	2.058	2.428	2.815	3.064	3.293	3.571	3.777	3.923
$m-19$	1.995	2.395	2.797	3.045	3.274	3.555	3.760	3.909
$m-20$	1.967	2.369	2.769	3.026	3.256	3.539	3.749	3.896
$m-21$	1.924	2.352	2.755	3.019	3.238	3.523	3.729	3.884
$m-22$	1.884	2.323	2.738	2.991	3.221	3.507	3.717	3.873
$m-23$	1.849	2.296	2.710	2.973	3.205	3.491	3.695	3.842
$m-24$	1.811	2.269	2.695	2.957	3.190	3.474	3.676	3.816
$m-25$	1.770	2.250	2.671	2.936	3.175	3.458	3.659	3.794
$m-26$	1.726	2.221	2.655	2.915	3.160	3.440	3.644	3.776
$m-27$	1.685	2.192	2.634	2.901	3.145	3.421	3.626	3.763
$m-28$	1.639	2.162	2.606	2.872	3.130	3.401	3.611	3.754
$m-29$	1.585	2.132	2.578	2.857	3.115	3.380	3.573	3.750
$m-30$	1.396	2.015	2.486	2.762	3.003	3.305	3.499	3.689
.	.	.	.	.	.	.	.	
1	1.396	2.015	2.486	2.762	3.003	3.305	3.499	3.689
$ln(m)$	3.912	4.605	5.521	6.215	6.908	7.824	8.517	9.210

Table H$(\rho = 0.5, \nu = \infty, q = 0.05)$

m	50	100	250	500	1000	2500	5000	10000
m	2.883	3.047	3.252	3.393	3.528	3.696	3.816	3.943
$m-1$	2.714	2.894	3.113	3.272	3.416	3.599	3.717	3.848
$m-2$	2.605	2.798	3.025	3.183	3.332	3.522	3.647	3.777
$m-3$	2.522	2.724	2.962	3.119	3.272	3.463	3.599	3.725
$m-4$	2.454	2.665	2.905	3.073	3.228	3.418	3.558	3.686
$m-5$	2.394	2.612	2.866	3.036	3.189	3.383	3.527	3.657
$m-6$	2.340	2.566	2.823	3.003	3.155	3.356	3.497	3.629
$m-7$	2.291	2.525	2.792	2.967	3.127	3.332	3.472	3.606
$m-8$	2.249	2.489	2.759	2.938	3.103	3.306	3.448	3.584
$m-9$	2.205	2.456	2.729	2.912	3.080	3.283	3.426	3.564
$m-10$	2.164	2.424	2.704	2.887	3.058	3.261	3.406	3.546
$m-11$	2.129	2.394	2.676	2.864	3.037	3.241	3.388	3.528
$m-12$	2.093	2.367	2.655	2.843	3.017	3.223	3.370	3.512
$m-13$	2.059	2.338	2.635	2.823	2.999	3.206	3.354	3.497
$m-14$	2.022	2.313	2.610	2.804	2.981	3.190	3.339	3.483
$m-15$	1.990	2.290	2.594	2.786	2.964	3.175	3.325	3.469
$m-16$	1.959	2.262	2.576	2.769	2.948	3.162	3.312	3.457
$m-17$	1.925	2.243	2.553	2.753	2.932	3.149	3.300	3.445
$m-18$	1.891	2.218	2.539	2.738	2.917	3.136	3.288	3.433
$m-19$	1.864	2.205	2.523	2.723	2.903	3.124	3.276	3.422
$m-20$	1.831	2.175	2.499	2.708	2.889	3.113	3.265	3.411
$m-21$	1.800	2.154	2.491	2.694	2.876	3.101	3.254	3.400
$m-22$	1.767	2.136	2.469	2.679	2.863	3.090	3.243	3.390
$m-23$	1.734	2.113	2.460	2.665	2.851	3.078	3.232	3.379
$m-24$	1.697	2.092	2.442	2.650	2.838	3.066	3.221	3.368
$m-25$	1.670	2.071	2.419	2.635	2.826	3.054	3.209	3.357
$m-26$	1.630	2.053	2.411	2.620	2.814	3.041	3.197	3.346
$m-27$	1.594	2.029	2.390	2.603	2.803	3.027	3.184	3.334
$m-28$	1.554	2.004	2.375	2.586	2.782	3.013	3.170	3.322
$m-29$	1.512	1.997	2.356	2.569	2.767	2.997	3.154	3.308
$m-30$	1.335	1.879	2.278	2.505	2.703	2.936	3.096	3.245
.	.	.	.	.	.	.	.	.
1	1.335	1.879	2.278	2.505	2.703	2.936	3.096	3.245
$ln(m)$	3.912	4.605	5.521	6.215	6.908	7.824	8.517	9.210

7. vii) Use of tables 2, 3 and 4 can make using step-down FDR procedures simple and easily manageable, though requiring some normality and correlation assumptions. The procedures control the FDR, and the number of false positives can be controlled by “truncation" choice. Critical values for parameters not given by the tables can be obtained using the Fortran program SEQDN.

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