Differential Equations?
Section 1.1

In college algebra you were asked to find the solution(s) to equations. The answers to most of these questions were numbers (e.g. solve $2x+5=7$ for x, solve $y=x^2-3x+2$ for y is $x=2$, etc.). In Calculus, you were primarily solving for the derivative or integral of a function. The result (answer) therefore was itself a function. The field of Differential Equations takes this to another level in that the equations we consider here have as entities various derivatives of some function. It will be our task to determine what that function is. In a simple form, you have already done this. Consider the first integrals you encountered, for example $\int 2x\,dx$. Prior to calling this an integral (for a very good reason), your text called this an *anti-derivative*. This was in part to help you look upon the integrand ($2x$) as the derivative of some function. Your task was to find that function and you should have arrived at the answer of $x^2+\mathrm{C}$ (remember the $+\mathrm{C}$?). If we formalize the notation of this elementary example we can see that the genesis of the types of "questions" with which we will be dealing. Let's start by using nice, formal, mathematics notation to "ask" the previous question. Seeking the "answer" to $\int 2x\,dx$ is the same as asking, "What function of x has a derivative of $2x$?". Hence our formal presentation looks like:

$y'=2x$	What function of x has a derivative of $2x$?
$\dfrac{dy}{dx}=2x$	"Prime notation" is short but can be vague. Leibnitz notation is more formal.
$dy=2x\,dx$	Separate the variables (i.e. put all x's on one side of the equation, etc.).
$\displaystyle\int dy=\int 2x\,dx$	Integrate both sides.
$y=2\dfrac{x^2}{2}+\mathrm{C}$	Evaluate the integrals (i.e. find the anti-derivatives).
$y(x)=2\dfrac{x^2}{2}+\mathrm{C}$	y is a function of x. That is $y=y(x)$.
$y(x)=x^2+\mathrm{C}\,\blacksquare$	The answer.

A reflection to the days of Calculus 2 (primarily) will remind you of the terror of the various ***techniques of integration*** (i.e. integration by parts, trigonometric substitutions, partial fractions, inverse trigonometric substitution, hyperbolic trigonometric substitution, etc.). All in all, the vast majority of Calculus 2 was dedicated to solving a single type of *differential equation*, namely $\frac{dy}{dx} = f(x)$. What varied was the form of the function $f(x)$.

What kind of modifications can we make to the equation, $\frac{dy}{dx} = f(x)$ without changing the $f(x)$? How about changing the first derivative into a second derivative, $\frac{d^2y}{dx^2} = 2x$ ($f(x) = 2x$, from our example)? That would seem straightforward ... we'd integrate twice. $y(x) = \frac{x^3}{3} + \mathrm{C}x + \mathrm{D}$. What if we really got creative? How would you go about solving, $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2x$? Now we've got an equation which has both the first and second derivatives of some function ... and we're supposed to find that function. You've just entered the world of differential equations...be afraid, be very afraid!

Examples:

$$\frac{dy}{dx} = 2x + y \tag{1}$$

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 15x = 0 \tag{2}$$

$$\frac{\partial^2 V}{\partial x^2} - 2\frac{\partial^2 V}{\partial y^2} = V \tag{3}$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E\omega\, cos\, \omega t \tag{4}$$

Note: What's different about equation #3 compared to the others?

Definitions
Section 1.2

From our previous discussion, it should be clear that the complexities of differential equations could conceivably be vast and easily overwhelm us. In order to narrow our focus to something we can handle (remember this is your first ODE course) we must categorize DE's so that we can determine what techniques to apply, if any. Consider the following 2 integrals, $\int xe^{x^2}dx$ and $\int e^{x^2}dx$. There may be only a "slight" difference in the two but the first is encountered early in calculus 2 whereas the second in calculus 3 (polar coordinate transformations). The same situation occurs in DE. There are a great many tools to use on differential equations but before we can apply a tool we must know which tool to use. That means we must identify the type of DE we have. The first term we will learn is called the order of the differential equation. It is defined to be the order of the highest occurring derivative in the differential equation.

Order:
The order of the highest occurring derivative in the differential equation.

Examples:

$$\frac{dy}{dx} = 2x + y \qquad \text{1st order}$$

$$\frac{d^3x}{dt^3} - 2\frac{dx}{dt} - 15x = 0 \qquad \text{3rd order}$$

$$\frac{\partial^2 V}{\partial x^2} - 2\left(\frac{\partial^2 V}{\partial y^2}\right)^3 = V \qquad \text{2nd order}$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E\omega\, cos\, \omega t \qquad \text{2nd order}$$

The n^{th}-order equation and its solution

In general, any differential equation which can be written as:

$$F\left(x, y, y', \cdots, y^{(n)}\right) = 0 \tag{1}$$

is called an "n^{th}-order" ordinary differential equation. We will assume for this course that all equations of this type can be solved explicitly for $y^{(n)}$ in terms of the other $(n+1)$ variables to obtain

$$y^{(n)} = f\left(x, y, y', \cdots, y^{(n-1)}\right) \tag{2}$$

A function ϕ, defined on an open interval $a < x < b$, is called the **solution** to the differential equation (2) provided that the n derivatives all exist on the interval and $\phi^{(n)}(x) = f\left(x, \phi(x), \phi'(x), \cdots, \phi^{(n-1)}(x)\right)$ for every value of x in the interval.

Examples:

Question:

Show that $\phi(x) = e^{-x} + x - 1$ is a solution to $\phi' + \phi = x$; $\phi(0) = 0$.

Answer:

•First Order DE: We will need the first derivative of our solution.

$$\rightarrow \qquad \phi' = -e^{-x} + 1$$

•We will need to show that $\phi(0) = 0$ $\rightarrow$

$$\begin{aligned} \phi(0) &= e^{-x} + x - 1 \\ &= e^{-0} + 0 - 1 \\ &= 0\checkmark \end{aligned}$$

•Now we must verify that our solution does in fact solve the DE:

$$\begin{aligned} \phi' + \phi &= x \\ (-e^{-x} + 1) + (e^{-x} + x - 1) &= x \\ -e^{-x} + 1 + e^{-x} + x - 1 &= x \\ -e^{-x} + 1 + e^{-x} + x - 1 &= x \\ x &= x\checkmark \end{aligned}$$

The proposed solution solves the DE and satisfies the initial condition (i.e. $\phi(0) = 0$).■

Question:

Show that $S(t) = 8cos(3t) + 6sin(3t)$ is a solution to $S'' = -9S; S(0) = 8,$ $y'(0) = 18.$

Answer:

•Second Order DE: We will need the first and second derivatives of our solution.

First Derivative: $S'(t) = -24sin(3t) + 18cos(3t)$

Second Derivative: $S''(t) = -72cos(3t) - 54sin(3t)$

•We must be sure our solution satisfies the initial conditions:

$S(0) = 8cos(0) + 6sin(0) = 8\checkmark \qquad S'(0) = -24sin(0) + 18cos(0) = 18\checkmark$

•Now we need to show it solves the DE:

$$\begin{aligned} S'' &= -9S \\ -72cos(3t) - 54sin(3t) &= -9[8cos(3t) + 6sin(3t)] \\ &= -72cos(3t) - 54sin(3t)\checkmark \end{aligned}$$

We have shown that the function is indeed a solution to the DE and satisfies the initial conditions■

Linear and Non-Linear DEs

Another characterization of a DE is whether or not it can be described as linear or non-linear. The concept of linearity is a profound one. It appears in many ways in a wide spectrum of mathematical fields.

An n^{th}-order **linear** DE can be written as:

$$b_o(x)\frac{d^ny}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + b_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x).$$

Though this may be a little imposing at first glance, close inspection should show you that essentially, a DE is said to be linear if:

- Derivatives of y appear "by themselves" with only functions of x as coefficients.
- Nowhere does a derivative of y appear raised to any power other than 1.
- The right hand side of the DE must be devoid of y's (indeed for nearly half of our course, the right side of the equation will be zero).

Partial/Ordinary Differential Equations

The last characterization for us to consider is whether a DE is categorized as *partial* or *ordinary*. A partial differential equation has partial derivatives in it whereas an ordinary differential equation does not (remember multivariate functions?).

We have discussed briefly the characterizations we will use to identify differential equations. It's appropriate that we get a little practice. The following chart shows some examples of DEs along with their characterizations:

DE	Order	Ordinary/Partial	Linear/Non-Linear
$(x^2+y^2)dx+2xdy=0$	1	Ordinary	Non-Linear
$y'+P(x)y=Q(x)$	1	Ordinary	Linear
$y'''-3y'+2y=0$	3	Ordinary	Linear
$yy''=x$	2	Ordinary	Non-Linear
$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$	2	Partial	Linear
$x\frac{\partial^2 y}{\partial t^2}-y\frac{\partial^2 y}{\partial t^2}=y-4$	2	Partial	Non-Linear

Families of Curves

Section 1.3

The solution to a differential equation (if any exist at all) may or may not be unique. That means there may be more than one solution to the DE. We have already seen that along with solving the DE we sometimes must satisfy initial conditions for that DE. If we solve for the solution to a DE and no initial conditions were specified, we generally end up not with a single function but rather with a family of functions each of which is a solution to the DE.

This situation is not really new to us. Calculus students are taught when evaluating an indefinite integral to include the " + C". This integration constant can take on any value unless further information is available to define it. Consider the following indefinite integral from Calculus :

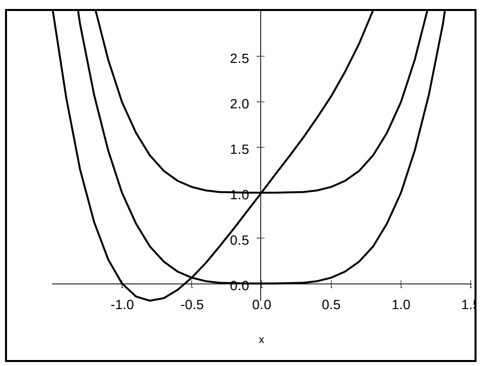
$$\int 3x^2\,dx = x^3 + C$$

This integral can be written as $\frac{dy}{dx} = 3x^2$. The solution, $y(x) = x^3 + C$ is an infinitely large family of cubic curves (MathCAD assignment), each of which is a solution to the DE (i.e. each of which have a derivative of $3x^2$).

Geometric Interpretation
Section 1.4

Since the solution to an indefinite integral is a family of curves, it is not surprising to find that the solution to a DE can also be a family of curves. The one-parameter family of curves in the above example satisfies an important property: *Through each point in the plane there passes one and only one member of the family of solutions*.

Now let's consider a second-order DE, $\frac{d^2y}{dx^2} = 12x^2$ whose solutions are $y = x^4 + c_1x + c_2$ (integrate twice). The plot below show the graph of 3 members of this family. Note that there are two solutions passing through the point $(0, 1)$ and 2 solutions passing through a point near $\left(\frac{-1}{2}, 0\right)$.



In the first-order case we had a unique solution once a point in space had been specified (i.e. We knew what the integration constant had to be for the curve to pass through that point). Here, in the second-order case, we cannot determine a unique solution just by specifying a point in space.

What if we specify not only a point in space but also the slope of our solution at that point? Examination of either point ($x = 0$ or $x = -\frac{1}{2}$) shows clearly that although there are 2 curves passing through each of those points, their slopes are not the same. Therefore, to determine a unique solution in this second-order case, we could specify a position in space through which our solution must pass as well as the slope of our solution at that point. Thus for the second-order case, our geometric argument has to be amended: *Through each point in the plane there passes one and only one member of the family of solutions that has a given slope*.

The Isoclines of an Equation

Section 1.5