Beam conditions for radiation generated by an electromagnetic Gaussian Schell-model source

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It was shown recently that the basic properties of a fluctuating electromagnetic beam can be derived from knowledge of a 2×2 cross-spectral density matrix of the electric field in the source plane. However, not every such matrix represents a source that will generate a beamlike field. We derive conditions that the matrix must satisfy for the source to generate an electromagnetic Gaussian Schell-model beam. © 2004 Optical Society of America

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It was recently shown that the changes in the propagation of the spectral density, the spectral degree of coherence, and the spectral degree of polarization of a fluctuating beam of electromagnetic radiation can be determined from knowledge of a 2×2 electric correlation matrix known as the cross-spectral density matrix of the electromagnetic beam.[1,2] This matrix has already been used in studies of the effects of the turbulent atmosphere on a beam propagating through it[3,4] and in investigations of the effects of liquid crystals and spatial light modulators on the degree of polarization of a fluctuating electromagnetic beam.[5]

In this Letter we derive the conditions that the cross-spectral density matrix of a planar, secondary, electromagnetic, Gaussian Schell-model source must satisfy to generate a beam. The results are generalizations of conditions derived previously (Ref. 6, Sec. 5.6.4) for the generation of such beams, within the framework of the scalar theory of coherence. These conditions are likely to be important for the design of diffusers that generate highly directional beams for various applications, such as guiding, aiming, and communications.

Consider a planar, secondary, fluctuating electromagnetic source, located in the plane $z = 0$ and radiating into half-space $z > 0$. We assume that the fluctuations are represented by a statistical ensemble that is stationary, at least in the wide sense. We will characterize the second-order correlation properties of the source by the 2×2 cross-spectral density matrix[1] defined as[7]

$$\overleftrightarrow{W}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \begin{bmatrix} W_{xx}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) & W_{xy}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) \\ W_{yx}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) & W_{yy}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) \end{bmatrix}. \tag{1}$$

Here $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are the two-dimensional position vectors of two points Q_1 and Q_2 in the source plane (see Fig. 1), ω denotes the frequency, and

$$W_{ij}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \langle E_i^*(\boldsymbol{\rho}_1; \omega) E_j(\boldsymbol{\rho}_2; \omega) \rangle, \quad (i = x, y; \quad j = x, y). \tag{2}$$

Here E_i and E_j denote the components of the electric field in two mutually orthogonal x and y directions perpendicular to the z axis, and the angle brackets denote an average taken over an ensemble of realizations of the electric field in the context of coherence theory in the space–frequency domain (Ref. 6, Sec. 4.7.1). We consider only beams generated by electromagnetic Gaussian Schell-model sources.[4,8] For such sources the elements of the $\overleftrightarrow{W}$ matrix have the form

$$W_{ij}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \sqrt{S_i^{(0)}(\boldsymbol{\rho}_1; \omega)} \sqrt{S_j^{(0)}(\boldsymbol{\rho}_2; \omega)} \times \eta_{ij}^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1; \omega), \quad (i = x, y; \quad j = x, y). \tag{3}$$

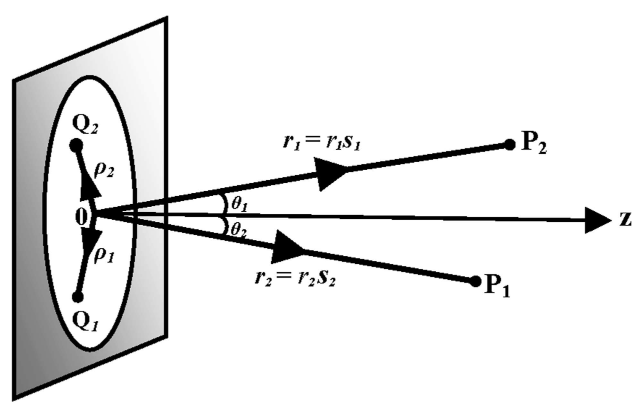

Fig. 1. Illustration of the notation used for the propagation of an electromagnetic beam.

In this formula $S_i^{(0)}$ and $S_j^{(0)}$ denote the spectral densities (proportional to the average electric energy densities) of the i and j components of the electric field in the plane $z = 0$, and $\eta_{ij}^{(0)}$ denote the degree of correlation between the two components. These quantities are given by expressions of the form[4]

$$S_j^{(0)}(\boldsymbol{\rho};\omega) = A_j^2 \exp\left(-\frac{\boldsymbol{\rho}^2}{2\sigma^2}\right), \qquad (j = x, y), \quad (4)$$

$$\eta_{ij}^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1;\omega) = B_{ij} \exp\left[-\frac{(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)^2}{2\delta_{ij}^2}\right], \quad (i = x, y; \quad j = x, y). \quad (5)$$

Parameters A_i, B_{ij}, σ, and δ_{ij} are independent of position but in general depend on frequency.[9]

Let us now determine the electric cross-spectral density matrix of the radiated electric field in the far zone at points P_1 and P_2 specified by position vectors $\mathbf{r}_1 = r_1\mathbf{s}_1$ and $\mathbf{r}_2 = r_2\mathbf{s}_2$ ($\mathbf{s}_1^2 = \mathbf{s}_2^2 = 1$). Following an argument similar to that used in scalar theory (Ref. 6, Sec. 5.3.1), we readily find that

$$W_{ij}^{(\infty)}(r_1\mathbf{s}_1, r_2\mathbf{s}_2;\omega) = (2\pi k)^2 \cos\theta_1 \cos\theta_2 \times \left\{\widetilde{W}_{ij}^{(0)}(-k\mathbf{s}_{1\perp}, k\mathbf{s}_{2\perp};\omega)\frac{\exp[ik(\mathbf{r}_2 - \mathbf{r}_1)]}{\mathbf{r}_1\mathbf{r}_2}\right\}, \quad (i = x, y; \quad j = x, y), \quad (6)$$

where

$$\widetilde{W}_{ij}^{(0)}(\mathbf{f}_1, \mathbf{f}_2;\omega) = \frac{1}{(2\pi)^4}\iint W_{ij}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2;\omega) \times \exp[-i(\mathbf{f}_1\cdot\boldsymbol{\rho}_1 + \mathbf{f}_2\cdot\boldsymbol{\rho}_2)]\mathrm{d}^2\boldsymbol{\rho}_1\mathrm{d}^2\boldsymbol{\rho}_2, \quad (i = x, y; \quad j = x, y), \quad (7)$$

is the four-dimensional Fourier transform of $W_{ij}^{(0)}$. The quantities $\mathbf{s}_{1\perp}$ and $\mathbf{s}_{2\perp}$ are projections, considered as two-dimensional vectors, of the unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ onto source plane $z = 0$, and θ_1 and θ_2 are the angles that unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ make with a positive z direction (see Fig. 1).

We see from Eq. (6) that to determine $\overleftrightarrow{W}^{(\infty)}$ we must first calculate the four-dimensional Fourier transform of matrix elements $W_{ij}^{(0)}$. One finds from Eqs. (7) and (3)–(5) that

$$\begin{aligned}\widetilde{W}_{ij}^{(0)}(\mathbf{f}_1, \mathbf{f}_2;\omega) &= \frac{A_iA_jB_{ij}}{(2\pi)^4}\iint \exp\left(-\frac{\boldsymbol{\rho}_1^2 + \boldsymbol{\rho}_2^2}{4\sigma^2}\right) \times \exp\left[-\frac{(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)^2}{2\delta_{ij}^2}\right] \times \exp[-i(\mathbf{f}_1\cdot\boldsymbol{\rho}_1 + \mathbf{f}_2\cdot\boldsymbol{\rho}_2)]\mathrm{d}^2\boldsymbol{\rho}_1\mathrm{d}^2\boldsymbol{\rho}_2 \\ &= \frac{A_iA_jB_{ij}}{(2\pi)^4}\iint \exp[-(a_{ij}\boldsymbol{\rho}_1^2 + a_{ij}\boldsymbol{\rho}_2^2 - 2b_{ij}\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)] \times \exp[-i(\mathbf{f}_1\cdot\boldsymbol{\rho}_1 + \mathbf{f}_2\cdot\boldsymbol{\rho}_2)]\mathrm{d}^2\boldsymbol{\rho}_1\mathrm{d}^2\boldsymbol{\rho}_2,\end{aligned}$$
$$(i = x, y; \quad j = x, y), \quad (8)$$

where

$$a_{ij} = \frac{1}{2}\left(\frac{1}{2\sigma^2} + \frac{1}{\delta_{ij}^2}\right), \qquad b_{ij} = \frac{1}{2\delta_{ij}^2}\cdot \quad (9)$$

The Fourier transform on the right-hand side of Eq. (5) can be readily evaluated, and one finds that

$$\widetilde{W}_{ij}^{(0)}(\mathbf{f}_1, \mathbf{f}_2;\omega) = \frac{A_iA_jB_{ij}}{(2\pi)^4(a_{ij}^2 - b_{ij}^2)} \times \exp[-(\alpha_{ij}\mathbf{f}_1^2 + \alpha_{ij}\mathbf{f}_2^2 + 2\beta_{ij}\mathbf{f}_1\cdot\mathbf{f}_2)], \quad (i = x, y; \quad j = x, y), \quad (10)$$

where

$$\alpha_{ij} = \frac{a_{ij}}{4(a_{ij}^2 - b_{ij}^2)}, \qquad \beta_{ij} = \frac{b_{ij}}{4(a_{ij}^2 - b_{ij}^2)}\cdot \quad (11)$$

On substituting from Eq. (10) into Eq. (6) we finally obtain the following expression for the elements of the cross-spectral density matrix of the electric field in the far zone generated by an electromagnetic Gaussian Schell-model source:

$$W_{ij}^{(\infty)}(r_1\mathbf{s}_1, r_2\mathbf{s}_2;\omega) = k^2\cos\theta_1\cos\theta_2\frac{A_iA_jB_{ij}}{(a_{ij}^2 - b_{ij}^2)} \times \exp[-2k^2(\alpha_{ij} - \beta_{ij}\mathbf{s}_1\cdot\mathbf{s}_2)] \times \frac{\exp[ik(\mathbf{r}_2 - \mathbf{r}_1)]}{\mathbf{r}_1\mathbf{r}_2}, \quad (i = x, y; \quad j = x, y). \quad (12)$$

The spectral density (more precisely, the average electric energy density) at a point P with position vector $\mathbf{r}$ ($\mathbf{r} = \mathbf{r}_1 = \mathbf{r}_2$) in the far zone is given by the expression

$$S^{(\infty)}(\mathbf{r};\omega) = \langle E_x^*(\mathbf{r},\omega)E_x(\mathbf{r},\omega)\rangle + \langle E_y^*(\mathbf{r},\omega)E_y(\mathbf{r},\omega)\rangle = \mathrm{Tr}[\overleftrightarrow{W}^{(\infty)}(\mathbf{r},\mathbf{r},\omega)] \quad (13)$$

and is readily evaluated by substituting from Eq. (12) into this formula. One then finds that

$$S^{(\infty)}(\mathbf{r};\omega) = \frac{k^2\cos^2\theta}{\mathbf{r}^2} \times \left\{\frac{A_x^2}{(a_{xx}^2 - b_{xx}^2)}\exp[-2k^2(\alpha_{xx} - \beta_{xx})] + \frac{A_y^2}{(a_{yy}^2 - b_{yy}^2)}\exp[-2k^2(\alpha_{yy} - \beta_{yy})]\right\}, \quad (14)$$

where $\theta = \theta_1 = \theta_2$. In order that the matrix $\overleftrightarrow{W}^{(0)}$ gives rise to a beam propagating close to a z axis, the spectral density $S^{(\infty)}(r\mathbf{s},\omega)$ must be negligible except when unit vector $\mathbf{s}$ lies in a narrow solid angle about the z axis. Then $\cos\theta$ can be approximated by unity, and we see from Eq. (14) that this will be the case if

$$\frac{1}{2(\alpha_{xx} - \beta_{xx})} \ll k^2, \qquad \frac{1}{2(\alpha_{yy} - \beta_{yy})} \ll k^2, \quad (15)$$

or in terms of source parameters

$$\frac{1}{4\sigma^2} + \frac{1}{\delta_{xx}^2} \ll \frac{2\pi^2}{\lambda^2}, \qquad \frac{1}{4\sigma^2} + \frac{1}{\delta_{yy}^2} \ll \frac{2\pi^2}{\lambda^2} \cdot \qquad (16)$$

These conditions are analogous to the beam conditions that were derived previously for scalar Gaussian Schell-model beams (Ref. 6, Sec. 5.6.4). We note that conditions (16) do not contain any information about the off-diagonal elements of the cross-spectral density matrix $\overleftrightarrow{W}^{(0)}$; i.e., they are independent of the state of polarization of the source.

In summary, we have derived constraints on the elements of the 2×2 electric cross-spectral density matrix of an electromagnetic Gaussian Schell-model source in order that the source generates a beam. These conditions are generalizations of conditions for the generation of scalar Gaussian Schell-model beams.

As mentioned earlier in this Letter, our results are likely to be important in connection with the design of diffusers that generate highly directional beams. Such beams are needed for many applications, for example, guiding, aiming, and communications.

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