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The far-zone behavior of the degree of polarization of electromagnetic beams propagating through atmospheric turbulence

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Abstract

It is shown analytically that the degree of polarization of a beam generated by an electromagnetic Gaussian Schell-model source which propagates through atmospheric turbulence tends to its value at the source plane with increasing distance of propagation. This result is independent of the spectral degrees of correlation of the source and of the strength of atmospheric turbulence. These conclusions are illustrated by a numerical example.

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1. Introduction

In recent years several investigations demonstrated that the degree of polarization of a partially coherent beam which propagates in free space changes as the beam propagates [1,2]. In this note we apply the recently developed unified theory of coherence and polarization [3–5] to derive an expression, valid at an arbitrary distance from the source, for the degree of polarization of a beam propagating through a turbulent atmosphere. The source is assumed to generate an electromagnetic generalization of the so-called Gaussian Schell-model beam, which is a well-known model of partially coherent beams used in many investigations. The usual model is based on scalar theory and it includes, as special case, some well-known laser modes. We show that after propagating a sufficiently long distance the degree of polarization, whilst changing with the distance of propagation, returns to its initial value (its value in the source plane), irrespective of the atmospheric conditions. Special case of this result has already been reported in the literature [6].

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2. The 2×2 cross-spectral density matrix of random Gaussian Schell-model source

The second-order coherence and polarization properties of a random, statistically stationary electromagnetic beam may be characterized by a 2×2 electric cross-spectral density matrix [3]

$$\overleftrightarrow{W}(\mathbf{r}_1, \mathbf{r}_2, \omega) \equiv W_{ij}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle E_i^*(\mathbf{r}_1, \omega) E_j(\mathbf{r}_2, \omega) \rangle \quad (i = x, y, \; j = x, y), \tag{2.1}$$

where $\mathbf{E} = (E_x, E_y)$ are members of a statistical ensemble representing a fluctuating electric field at a point $\mathbf{r}$, at frequency ω and the angular brackets denote the average taken over the ensemble of realizations of the electric field in the sense of the coherence theory in the space-frequency domain (cf. [7], Section 4.7.1). The E_x and E_y components are taken along two mutually orthogonal directions at right angle to the direction of propagation of the beam (the z-direction).

Since we are interested in propagation close to the z-direction it is convenient to set $\mathbf{r} \equiv (\boldsymbol{\rho}, z)$, where $\boldsymbol{\rho}$ is a two-dimensional vector perpendicular to the beam axis and z is the distance from the source plane (see Fig. 1).

The elements of the cross-spectral density matrix in the source plane may be expressed in the form

$$W_{ij}^{(0)}(\boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2, \omega) = \sqrt{S_i^{(0)}(\boldsymbol{\rho}'_1, \omega)} \sqrt{S_j^{(0)}(\boldsymbol{\rho}'_2, \omega)} \times \eta_{ij}^{(0)}(\boldsymbol{\rho}'_2 - \boldsymbol{\rho}'_1, \omega) \quad (i = x, y, \; j = x, y), \tag{2.2}$$

where $S_i^{(0)}$ represents the spectral density of the component E_i of the electric field and $\eta_{ij}^{(0)}$ is the spectral degree of correlation between the components E_i and E_j in the plane $z = 0$. These quantities may be determined experimentally [5]. In Appendix A we show that for all values of their argument and all pairs of the indexes i, j the spectral degrees of correlation satisfy the inequality $|\eta_{ij}^{(0)}| \leqslant 1$.

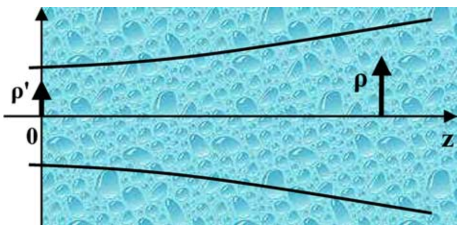

Fig. 1. Illustrating the notation relating to propagation of an electromagnetic beam through a turbulent atmosphere.

We will consider random electromagnetic beams generated by Gaussian Shell-model sources. For such sources [2]

$$S_i^{(0)}(\boldsymbol{\rho}', \omega) = A_i^2 \exp(-\boldsymbol{\rho}'^2 / 2\sigma_i^2) \quad (i = x, y), \tag{2.3}$$

$$\eta_{ij}^{(0)}(\boldsymbol{\rho}'_2 - \boldsymbol{\rho}'_1, \omega) = B_{ij} \exp\left[-\frac{(\boldsymbol{\rho}'_2 - \boldsymbol{\rho}'_1)^2}{2\delta_{ij}^2} \right] \quad (i = x, y, \; j = x, y). \tag{2.4}$$

In these expressions the factors A_i and B_{ij} are independent of position but may depend on frequency. The same is true about the variances σ_i and δ_{ij}. Moreover, the factor B_{ij} has the following properties:

$$B_{ij} \equiv 1 \quad \text{when } i = j, \tag{2.5a}$$

$$|B_{ij}| \leqslant 1 \quad \text{when } i \neq j, \tag{2.5b}$$

$$B_{ji} = B_{ij}^*, \tag{2.5c}$$

asterisk denoting the complex conjugate. Eq. (2.5a) follows from the fact that when $j = i$ $\eta_{ij}^{(0)}$ is just the usual spectral degree of coherence of the scalar theory (cf. [7], Section 4.3.2), which is well known to have the value unity when its two spatial arguments coincide. The inequality (2.5b) follows from the fact that $|\eta_{ij}^{(0)}| \leqslant 1$ as we already noted. The relation (2.5c) follows from Eq. (2.4) and the relation $W_{ji}(\mathbf{r}_1, \mathbf{r}_2, \omega) = W_{ij}^*(\mathbf{r}_2, \mathbf{r}_1, \omega)$ that is an immediate consequence of the definition of the matrix $\overleftrightarrow{W}$. We will now derive an expression for the degree of polarization of a Gaussian–Schell model beam propagating through the turbulent atmosphere.

[2] Properties of beams generated by such sources have been extensively studied by Gori et al. [2].

3. Degree of polarization of the Gaussian Schell-model beam in a turbulent atmosphere

The spectral degree of polarization of the beam at a point $\boldsymbol{r}$ is given by the expression (cf. [3], Eq. (11))

$$\mathscr{P}(\boldsymbol{r},\omega)=\sqrt{1-\frac{4\,\mathrm{Det}\,\overleftrightarrow{W}(\boldsymbol{r},\boldsymbol{r};\omega)}{\left[\mathrm{Tr}\,\overleftrightarrow{W}(\boldsymbol{r},\boldsymbol{r};\omega)\right]^2}}, \tag{3.1}$$

where $\mathrm{Det}\,\overleftrightarrow{W}$ and $\mathrm{Tr}\,\overleftrightarrow{W}$ denote the determinant and the trace, respectively, of the matrix $\overleftrightarrow{W}$. More explicitly, in terms of the elements of the $\overleftrightarrow{W}$-matrix one readily finds that the degree of polarization is given by the formula

$$\mathscr{P}(\boldsymbol{r},\omega)=\frac{\sqrt{(W_{xx}-W_{yy})^2+4W_{xy}W_{yx}}}{W_{xx}+W_{yy}}, \tag{3.2}$$

where the arguments of all the elements of the $\overleftrightarrow{W}$-matrix are, of course, $(\boldsymbol{r},\boldsymbol{r};\omega)$.

To determine the degree of polarization across an electromagnetic Gaussian Schell-model source we must first determine the elements of the matrix $W_{ij}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$. From Eqs. (2.2) and (2.4) it follows at once that they are given by the expression

$$W_{ij}^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2',\omega)=A_iA_jB_{ij}\exp\left[-\left(\frac{\boldsymbol{\rho}_1'^2}{4\sigma_i^2}+\frac{\boldsymbol{\rho}_2'^2}{4\sigma_j^2}\right)\right]\times\exp\left[-\frac{(\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1')^2}{\delta_{ij}^2}\right]\quad(i=x,y,\ j=x,y). \tag{3.3}$$

To simplify the analysis we will assume that

$$\sigma_i=\sigma_j\equiv\sigma. \tag{3.4}$$

The formula (3.3) then reduces to

$$W_{ij}^{(0)}(\boldsymbol{\rho}',\boldsymbol{\rho}',\omega)=A_iA_jB_{ij}\exp\left[-\left(\frac{\boldsymbol{\rho}'^2}{2\sigma^2}\right)\right]\quad(i=x,y,\ j=x,y). \tag{3.5}$$

On substituting from Eq. (3.5) into the formula (3.2) with $\boldsymbol{r}\equiv\boldsymbol{\rho}$ and using Eq. (2.5c) we obtain for the degree of polarization across the source plane the expression

$$\mathscr{P}^{(0)}(\boldsymbol{\rho}',\omega)=\frac{\sqrt{(A_x^2-A_y^2)^2+4A_x^2A_y^2|B_{xy}|^2}}{A_x^2+A_y^2}. \tag{3.6}$$

It is to be noted that as a consequence of the simplifying assumption (3.4) the degree of polarization across the source plane is independent of the position vector $\boldsymbol{\rho}$, i.e., it is uniform across it. It is also of interest to note that apart from the correlation coefficient B_{xy}, the degree of polarization (3.6) in the source plane depends only on the ratio A_x/A_y, not on the actual values of A_x and A_y.

In a recent paper [6] an expression was derived for a cross-spectral density matrix of a electromagnetic Gaussian Schell-model beam at any point $\boldsymbol{r}\equiv(\boldsymbol{\rho},z)$ in the half space $z>0$ containing turbulent atmosphere. The Gaussian Schell-model source which generated the beam was somewhat restricted in that it was assumed that the off-diagonal elements $W_{xy}^{(0)}(\boldsymbol{\rho}',\omega)=W_{yx}^{(0)}(\boldsymbol{\rho}',\omega)=0$, i.e., that the x and y components of the electric field were uncorrelated at each source point. However, it is not difficult to generalize the analysis by removing this assumption and one then readily finds by a similar argument as used in [6] that

$$W_{ij}(\boldsymbol{\rho},\boldsymbol{\rho},z,\omega)=\frac{A_iA_jB_{ij}}{\Delta_{ij}^2(z)}\exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma^2\Delta_{ij}^2(z)}\right]\quad(i=x,y,\ j=x,y), \tag{3.7}$$

where $\Delta_{ij}^2(z)$ is the so-called effective beam spread. This quantity depends on the model which one uses for the atmospheric turbulence but, with the commonly used models, it has the form

$$\Delta_{ij}^2(z)=1+\alpha_{ij}z^2+Tz^m\quad(i=x,y,\ j=x,y), \tag{3.8}$$

where

$$\alpha_{ij}=\frac{1}{(k\sigma)^2}\left(\frac{1}{4\sigma^2}+\frac{1}{\delta_{ij}^2}\right). \tag{3.9}$$

In Eq. (3.8) T and m are parameters depending on the atmospheric model. The formula (3.8) shows that the spreading of the beam depends on both the source parameters (σ,δ_{ij}) and on the parameters describing the model used for the turbulence (T,m).

Two expressions are currently available for the beam spread. One is based on Tatarskii model of the spectrum of atmospheric fluctuations and the other on the Kolmogorov spectrum (cf. [8], Section 3.3). For the former model (Tatarskii)

$$T = 1.093 C_n^2 l_0^{-1/3} \sigma^{-2}, \quad m = 3, \tag{3.10}$$

and for the other model (Kolmogorov)

$$T = 0.98 (C_n^2)^{6/5} k^{2/5} \sigma^{-2}, \quad m = \frac{16}{5}. \tag{3.11}$$

In these expressions $k = \omega/c$ is the wave number of the light beam, C_n^2 is the refractive index structure parameter and l_0 is the inner scale of turbulence.

Expression for the beam spread $\Delta_{ij}^2(z)$ based on these two formulas has been derived in [9,10], using, however, the scalar theory rather than electromagnetic theory.

Based on the definition (3.1) for the degree of polarization, together with Eq. (3.7), we derive a general expression for the degree of polarization $\mathscr{P}(\boldsymbol{\rho}, z, \omega)$ of a Gaussian Schell-model beam propagating in atmospheric turbulence. We find that

$$\mathscr{P}(\boldsymbol{\rho}, z, \omega) = \frac{[F(z)]^{1/2}}{G(z)}, \tag{3.12}$$

where

$$F(z) = \left(\frac{A_x^2}{\Delta_{xx}^2(z)} \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma\Delta_{xx}^2(z)} \right] - \frac{A_y^2}{\Delta_{yy}^2(z)} \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma\Delta_{yy}^2(z)} \right] \right)^2 + \frac{4A_x^2 A_y^2 |B_{xy}|^2}{\Delta_{xy}^2(z)} \exp\left[-\frac{\boldsymbol{\rho}^2}{\sigma\Delta_{xy}^2(z)} \right], \tag{3.13}$$

$$G(z) = \frac{A_x^2}{\Delta_{xx}^2(z)} \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma\Delta_{xx}^2(z)} \right] + \frac{A_y^2}{\Delta_{yy}^2(z)} \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma\Delta_{yy}^2(z)} \right]. \tag{3.14}$$

4. Far-zone behavior of the degree of polarization of a beam propagating in a turbulent atmosphere

Based on the Tatarskii model one readily finds that after sufficiently long propagation distance the elements of cross-spectral density matrix of the Gaussian Schell-model beam are given by the expression

$$W_{ij}(\boldsymbol{\rho}, z, \omega) \sim \frac{A_i A_j B_{ij}}{T} z^{-3} - \frac{A_i A_j B_{ij} \alpha_{ij}}{T^2} z^{-4} + \mathrm{O}(z^{-5}) \tag{4.1}$$

as $kz \to \infty$, O denoting the order of magnitude symbol.

On substituting from Eq. (4.1) into the expression (3.12) for the degree of polarization one finds that as $kz \to \infty$

$$\mathscr{P}(\boldsymbol{\rho}, z, \omega) \to \frac{[F(z)]^{1/2}}{G(z)}, \tag{4.2}$$

where

$$F(z) = \left[\frac{A_x^2 - A_y^2}{T} z^{-3} - \frac{A_x^2 \alpha_{xx} - A_y^2 \alpha_{yy}}{T^2} z^{-4} + \mathrm{O}(z^{-5}) \right]^2 + 4 \left[\frac{A_x^2 A_y^2 B_{xy}}{T} z^{-3} - \frac{A_x^2 A_y^2 B_{xy}}{T^2} z^{-4} + \mathrm{O}(z^{-5}) \right]^2, \tag{4.3}$$

$$G(z) = \frac{A_x^2 + A_y^2}{T} z^{-3} - \frac{A_x^2 \alpha_{xx} + A_y^2 \alpha_{yy}}{T^2} z^{-4} + \mathrm{O}(z^{-5}). \tag{4.4}$$

It is clear from the formula (4.2), together with Eqs. (4.3) and (4.4), that the leading term (the term in the power of z^{-3}) does not depend on the coefficients for the effective beam spread α_{ij}, defined by Eq. (3.9). Moreover the "atmospheric coefficient" T cancels out in the expression (4.2) in the limit as $kz \to \infty$. Hence the degree of polarization, given by the formula (4.2), becomes

$$\mathscr{P}(\boldsymbol{\rho}, z, \omega) \sim \frac{\sqrt{(A_x^2 - A_y^2)^2 + 4A_x^2 A_y^2 |B_{xy}|^2}}{A_x^2 + A_y^2} \quad \text{as } kz \to \infty. \tag{4.5}$$

On comparing Eq. (4.5) with the expression (3.6) we see that

$$\mathscr{P}(\boldsymbol{\rho}, z, \omega) \sim \mathscr{P}^0(\boldsymbol{\rho}, \omega) \quad \text{as } kz \to \infty \tag{4.6}$$

the expressions on both sides of this formula being actually independent of the transverse variables ($\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$). This result was obtained on the basis of the Tatarskii model of turbulence. If instead one considers the Kolmogorov model one finds in place of Eq. (4.1) that the elements of the cross spectral density matrix are

$$W_{ij}(\boldsymbol{\rho}, z, \omega) = \frac{A_i A_j B_{ij}}{T} z^{-16/5} - \frac{A_i A_j B_{ij} \alpha_{ij}}{T^2} z^{-22/5} + \mathrm{O}(z^{-28/5}) \quad (i = x, y,\ j = x, y). \tag{4.7}$$

By similar arguments as based on Eq. (4.1) one finds that the conclusion expressed by Eq. (4.6) again holds.

In free space the expression for the degree of polarization of a beam sufficiently far away from the source plane can be derived from the general formula (3.12) by setting for the beam spread Δ_{ij} the free-space value

$$\mathscr{P}(\boldsymbol{\rho}, z, \omega) \sim \frac{\sqrt{\alpha_{xy}^2 \alpha_{yy}^2 A_x^2 - 2\alpha_{xx}\alpha_{yy}\alpha_{xy}^2 A_x A_y + \alpha_{xy}^2 \alpha_{xx}^2 A_y^2 + 4\alpha_{xx}^2 \alpha_{yy}^2 A_x^2 A_y^2 |B_{xy}|^2}}{(A_x^2 \alpha_{yy} + A_y^2 \alpha_{xx})\alpha_{xy}} \quad \text{as } kz \to \infty, \tag{4.8}$$

where the α_{ij} are given by the formula (3.9). In this case $\mathscr{P}(\boldsymbol{\rho}, z \to \infty, \omega) \neq \mathscr{P}(\boldsymbol{\rho}, z \to 0, \omega)$ and in general: the far-zone value of the degree of polarization depends on all the parameters of the source.

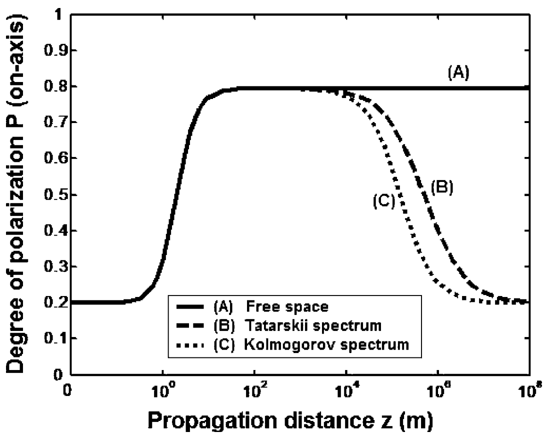

Fig. 2. The change of the degree of polarization of a Gaussian–Schell model beam propagating in a turbulent atmosphere, calculated from Eq. (3.2). The parameters characterizing the source are $\omega = 3 \times 10^{15}$ rad/sec ($\lambda = 0.628$ μm), $A_x^2 = A_y^2 = 0.5$, $B_{xy} = 0.2$, $\sigma_x = \sigma_y = \sigma = 5$ cm, $\delta_{xx} = \delta_{yy} = 0.1$ mm. The parameters characterizing the atmosphere were chosen to be $C_n^2 = 10^{-13} m^{2/3}$, $l_0 = 5$ mm.

The result expressed by Eq. (4.6) is the main conclusion of our analysis. It shows that after a sufficiently long distance of propagation through the atmosphere the degree of polarization of the beam returns to its initial value (its value in the source plane). Moreover, since the expression (4.5) depends on the A and B coefficients, the degree of polarization of the beam after it traveled over a sufficiently long distance depends neither on the spectral degrees of correlation η_{ij} of the Gaussian–Schell model source nor on the atmospheric turbulence. This conclusion is in agreement with a result derived in [6] for a more restricted class of beams.

In Fig. 2 we illustrate the behavior of the degree of polarization of electromagnetic Gaussian–Schell model beam both for propagation in turbulence and in free space with increasing distance from the source plane, for selected values of the parameters. The figure shows that for propagation in free space the degree of polarization acquires a particular value after propagating at a certain distance ($z = z_0$ say) and it retains this value as the beam propagates further, i.e., for $z > z_0$. On the other hand in turbulent atmosphere the degree of polarization returns to its initial value (the value it has in the source plane) after it propagates over a sufficiently long distance.

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Appendix A. Proof of the inequality $|\eta_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)| \leqslant 1$

We have the obvious inequality

$$\left\langle |a_1 E_i(\boldsymbol{r}_1, \omega) + a_2 E_j(\boldsymbol{r}_2, \omega)|^2 \right\rangle \geqslant 0, \tag{A.1}$$

where a_1 and a_2 are arbitrary constants. This inequality implies that

$$\begin{aligned} &a_1^* a_1 \langle E_i^*(\boldsymbol{r}_1, \omega) E_i(\boldsymbol{r}_1, \omega) \rangle \\ &\quad + a_2^* a_2 \left\langle E_j^*(\boldsymbol{r}_1, \omega) E_j(\boldsymbol{r}_2, \omega) \right\rangle \\ &\quad + a_1^* a_2 \langle E_i^*(\boldsymbol{r}_1, \omega) E_j(\boldsymbol{r}_2, \omega) \rangle \\ &\quad + a_1 a_2^* \left\langle E_j^*(\boldsymbol{r}_2, \omega) E_i(\boldsymbol{r}_2, \omega) \right\rangle \geqslant 0. \end{aligned} \tag{A.2}$$

We may express (A.2) in the form

$$\begin{aligned} &a_1^* a_1 S_i(\boldsymbol{r}_1, \omega) + a_2^* a_2 S_j(\boldsymbol{r}_2, \omega) + a_1^* a_2 W_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) \\ &\quad + a_1 a_2^* W_{ji}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) \geqslant 0. \end{aligned} \tag{A.3}$$

It follows from the definition of the off-diagonal elements of the cross-spectral density matrix that $W_{ji}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = W_{ij}^*(\boldsymbol{r}_2, \boldsymbol{r}_1, \omega)$. Using this fact and a well-known property of non-negative definite quadratic forms [11] it follows that the determinant

$$\begin{vmatrix} S_i(\boldsymbol{r}_1, \omega) & W_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) \\ W_{ij}^*(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) & S_j(\boldsymbol{r}_2, \omega) \end{vmatrix} \geqslant 0 \tag{A.4}$$

implying that

$$|\eta_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)| \equiv \frac{|W_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)|}{\sqrt{S_i(\boldsymbol{r}_1, \omega)}\sqrt{S_j(\boldsymbol{r}_2, \omega)}} \leqslant 1 \tag{A.5}$$

for all values of the arguments and of the suffixes.

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