

## THE ECHELON FORM

The echelon form of the augmented matrix  $(\mathbf{A}|\mathbf{b})$  is obtained by performing a sequence of elementary row operations. Ones are placed in the  $i^{\text{th}}$  row and  $i^{\text{th}}$  column element and zeros below. The echelon form results when the procedure terminates. The echelon form reveals the nature of the solutions to the original system of equations  $\mathbf{A}\underline{x} = \underline{b}$  where  $\mathbf{A}$  is the  $m$  by  $n$  coefficient matrix. Several examples follow.

Example 1 A system of 4 equations in 3 unknowns ( $m=4, n=3$ )

$$\begin{array}{cccccc} x & + & y & + & z & = & 0 \\ 2x & - & y & - & z & = & 3 \\ 3x & - & 2y & + & z & = & 2 \\ x & + & 4y & + & 2z & = & -1 \end{array}$$

$$\begin{aligned} (\mathbf{A}|\mathbf{b}) &= \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 3 & -2 & 1 & 2 \\ 1 & 4 & 2 & -1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & -5 & -2 & 2 \\ 0 & 3 & 1 & -1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -5 & -2 & 2 \\ 0 & 3 & 1 & -1 \end{array} \right) \\ &\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -2 & 2 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{aligned}$$

The final matrix is the echelon form of the augmented matrix. The original system has reduced to a consistent system of 3 equations with 3 unknowns. Hence, one of the 4 original equations was redundant (linearly dependent) and the resulting 3 by 3 system has a unique solution. The solution can be obtained by backward substitution in the 3 linearly independent equations obtained from the echelon form as in a Gauss Elimination. Alternatively, the Gauss-Jordan method can be used to insert zeros above the ones resulting in the identity matrix in the first 3 columns and the solution in the 4<sup>th</sup> column. Both methods are illustrated below.

I. Gauss Elimination

$$\begin{array}{lll} z = -1 & & \text{from the 3rd equation} \\ y + z = -1 \Rightarrow y = 0 & & \text{from the 2nd equation} \\ x + y + z = 0 \Rightarrow x = 1 & & \text{from the 1st equation} \end{array}$$

## II. Gauss-Jordan

$$(\mathbf{A}|\mathbf{b}) = \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right) \Rightarrow x = 1, y = 0, z = -1$$

There is a convenient matrix function in MATLAB which converts a matrix to its row reduced echelon form. This is similar to the echelon form discussed here except that zeros appear above as well as below the ones in the respective columns. It is actually more convenient to have the echelon form expressed in this way because the solution(s) to the system of equations become more apparent.

In Example 1 the echelon form revealed a consistent system with a unique solution. Additional work was required to find the solution. If we utilize the MATLAB function “rref” to obtain the row reduced echelon form of  $(\mathbf{A}|\mathbf{b})$ , the solution is obtained at the same time.

A =

```
1  1  1  0
2 -1 -1  3
3 -2  1  2
1  4  2 -1
```

EDU» rref(A)

ans =

```
1  0  0  1
0  1  0  0
0  0  1 -1
0  0  0  0
```

The nonzero rows of the row reduced echelon form are identical to those in the matrix obtained by the Gauss-Jordan method.

Example 2 A system of 5 equations in 3 unknowns (m=5, n=3)

$$\begin{array}{rccccrcr}
 x_1 & + & x_2 & + & x_3 & = & 0 \\
 2x_1 & - & x_2 & - & x_3 & = & 3 \\
 & & 2x_2 & - & x_3 & = & 4 \\
 3x_1 & - & 2x_2 & + & x_3 & = & -1 \\
 4x_1 & - & 3x_2 & + & x_3 & = & -1
 \end{array}$$

The echelon form of the augmented matrix is found the same way as in Example 1.

$$(\mathbf{A}|\mathbf{b}) = \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 0 & 2 & -1 & 4 \\ 3 & -2 & 1 & -1 \\ 4 & -3 & 1 & -1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -5 & -2 & -1 \\ 0 & -7 & -3 & -1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 4 & -8 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The echelon form of the augmented matrix indicates that 2 of the original 5 equations were redundant and the remaining 3 equations comprise a consistent system with a unique solution.

Example 3 A system of 6 equations in 3 unknowns (m=6, n=3)

$$\begin{array}{rccccrcr}
 x_1 & + & x_2 & + & x_3 & = & 0 \\
 2x_1 & - & x_2 & - & x_3 & = & 3 \\
 & & 2x_2 & - & x_3 & = & 4 \\
 3x_1 & - & 2x_2 & + & x_3 & = & -1 \\
 4x_1 & - & 3x_2 & + & x_3 & = & -1 \\
 10x_1 & - & 3x_2 & + & x_3 & = & 4
 \end{array}$$

The augmented matrix and its echelon form are given below.

$$(\mathbf{A}|\mathbf{b}) = \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 0 & 2 & -1 & 4 \\ 3 & -2 & 1 & -1 \\ 4 & -3 & 1 & -1 \\ 10 & -3 & 1 & 4 \end{array} \right) \sim \dots \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right)$$

In this case the echelon form implies the original system of equations was inconsistent as a result of the last line, which in equation form reads  $0x + 0y + 0z = -1$ . In fact, the system of equations in Example 3 are the same as in Example 2 with one additional equation. The left side of the added equation is simply the sum of the left side of all the equations above it. For the new system of equations to be consistent, it follows that the right hand side constant in the last equation must also be the sum of the constants in the original 5 equations. However; it is not and therefore the complete original system of equations is inconsistent.

Example 4 A system of  $m = 6$  equations in  $n = 5$  unknowns

$$a + b + c - d - 3e = 3$$

$$a - b - c - 3d + e = -1$$

$$2b + c + d - 2e = 4$$

$$a + 2c - 5e = 1$$

$$2a - b + c - 3d - 4e = 0$$

$$3a - 2b - 6d - 3e = -1$$

$$\begin{aligned}
 (\mathbf{A}|\mathbf{b}) &= \begin{pmatrix} 1 & 1 & 1 & -1 & -3 & 3 \\ 1 & -1 & -1 & -3 & 1 & -1 \\ 0 & 2 & 1 & 1 & -2 & 4 \\ 1 & 0 & 2 & 0 & -5 & 1 \\ 2 & -1 & 1 & -3 & -4 & 0 \\ 3 & -2 & 0 & -6 & -3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & -1 & -3 & 3 \\ 0 & -2 & -2 & -2 & 4 & -4 \\ 0 & 2 & 1 & 1 & -2 & 4 \\ 0 & -1 & 1 & 1 & -2 & -2 \\ 0 & -3 & -1 & -1 & 2 & -6 \\ 0 & -5 & -3 & -3 & 6 & -10 \end{pmatrix} \\
 &\sim \begin{pmatrix} 1 & 1 & 1 & -1 & -3 & 3 \\ 0 & 1 & 1 & 1 & -2 & 2 \\ 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 2 & 2 & -4 & 0 \\ 0 & 0 & 2 & 2 & -4 & 0 \\ 0 & 0 & 2 & 2 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & -1 & -3 & 3 \\ 0 & 1 & 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

In this example, the echelon form has reduced to an equivalent system with 3 equations and 5 unknowns. Since the last three rows consist entirely of zeros, the original

system of equations is consistent and equivalent to a system of 3 equations in 5 unknowns. In this case, two of the five variables may be chosen arbitrarily. The criterion for selecting which variables may be arbitrary is discussed later. Regardless of that decision, an infinite number of solutions are possible due to the existence of arbitrary unknowns.

One additional example is included to illustrate the case where there are fewer equations than unknowns.

Example 5 A system of 4 equations in 5 unknowns ( $m=4, n=5$ )

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 + 3x_5 &= 3 \\ 2x_1 - x_2 + x_3 - 2x_4 - 2x_5 &= 2 \\ x_2 - x_3 + 2x_4 + 2x_5 &= 0 \\ -x_1 + 3x_2 - x_3 + 3x_4 + 7x_5 &= 1 \end{aligned}$$

The row reduced echelon form from MATLAB is given below.

$$\text{rref}(\mathbf{A}|\mathbf{b}) = \left( \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0.5 & 2.5 & 1 \\ 0 & 0 & 1 & -1.5 & -0.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Once again the equations are consistent; however, one of the original equations was redundant. Similar to Example 4, a pair of arbitrary variables can be selected from the set  $(x_1, x_2, x_3, x_4, x_5)$ . As a result, an infinite number of solutions exist.

These previous examples illustrate the rationale for transforming the augmented matrix representing the system of equations into an echelon form. The echelon form is also useful in other ways. Example 6 demonstrates another useful application of the echelon form.

Example 6 Find the value(s) of  $K$  for which the system of equations are consistent.

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 2 \\
2x_1 - x_2 + x_3 &= 3 \\
3x_2 - 2x_3 + 3x_4 &= 1 \\
3x_1 - 3x_2 + 4x_3 - 2x_4 &= K
\end{aligned}$$

Before we look at the echelon form of the augmented matrix, its a good idea to check the coefficient matrix. For the 4 by 4 system denoted by  $\mathbf{A}\mathbf{x} = \mathbf{b}$ , if  $|\mathbf{A}|$  were other than zero, the matrix inverse  $\mathbf{A}^{-1}$  would exist as would a unique solution for any value of K. Consequently, the equations would be consistent for all values of K. After confirming that  $|\mathbf{A}| = 0$ , we turn to the echelon form for the answer.

$$\begin{aligned}
(\mathbf{A}|\mathbf{b}) &= \left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & 0 & 3 \\ 0 & 3 & -2 & 3 & 1 \\ 3 & -3 & 4 & -2 & K \end{array} \right) \sim \left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & -2 & -1 \\ 0 & 3 & -2 & 3 & 1 \\ 0 & -6 & 1 & -5 & K-6 \end{array} \right) \\
&\sim \left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 3 & -1 & K-4 \end{array} \right) \sim \left( \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & K-4 \end{array} \right)
\end{aligned}$$

A consistent system of equations with infinite solutions results when the last row of the echelon form is all zeros. Therefore, the only solution is  $K = 4$ .