ARBITRARY UNKNOWNS

The echelon form of the augmented matrix confirms the existence of arbitrary unknowns, i.e. a consistent system of equations in which one or more variables can be chosen arbitrarily There are several ways to establish if indeed a certain variable can be included in the subset of arbitrary unknowns. A few simple examples illustrate the point.

Example 1 For the system of equations below, establish the existence of 1 arbitrary unknown and determine if *x, y,* and *z* can each be arbitrary.

$$\begin{aligned} x + y + z &= 0 \\ 2x - y - z &= 3 \\ x + 4y + 4z &= -3 \\ x - 2y - 2z &= 3 \end{aligned}$$

$$(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 1 & 4 & 4 & -3 \\ 1 & -2 & -2 & 3 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & 3 & 3 & -3 \\ 0 & -3 & -3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The 4 by 4 system has been reduced to a 2 by 3 system. That is, there are only 2 linearly independent equations in the 3 variables (unknowns) and hence one of the variables is arbitrary. These equations are

$$\begin{aligned} x + y + z &= 0 \\ y + z &= -1 \end{aligned}$$

Now, suppose we try to make *z* the arbitrary unknown. This requires that solutions for *x* and *y* be expressed in terms of *z*. This can be attempted in one of two ways. One approach is to simply transfer all terms involving *z* over to the right side of the equations and consider *z* as a parameter. This yields

$$\begin{aligned} x + y &= -z \\ y &= -1 - z \end{aligned}$$

All that remains is substituting $y = -1 - z$ into the first equation and then solving for x. The final solution can be expressed as

$$x = 1, \quad y = -1-z, \quad z = \text{arbitrary}$$

Clearly, an infinite number of solutions exist since there is a different solution for each arbitrarily assigned value for z. A slightly different approach involves reformulating the reduced equations from the echelon form as

$$\hat{\mathbf{A}}\underline{\hat{\mathbf{x}}} = \underline{\hat{\mathbf{b}}} \quad \text{where} \quad \hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \underline{\hat{\mathbf{b}}} = \begin{pmatrix} -z \\ -1-z \end{pmatrix}, \quad \underline{\hat{\mathbf{x}}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. as a new system in matrix form with modified coefficient matrix $\hat{\mathbf{A}}$, constant vector $\underline{\hat{\mathbf{b}}}$, and vector of unknowns $\underline{\hat{\mathbf{x}}}$. The identical solution as given above is easily obtained from the modified augmented matrix

$$\left(\hat{\mathbf{A}}|\underline{\hat{\mathbf{b}}}\right) = \left(\begin{array}{cc|c} 1 & 1 & -z \\ 0 & 1 & -1-z \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1-z \end{array}\right)$$

$$x = 1, \quad y = -1-z, \quad z = \text{arbitrary}$$

The second approach is somewhat more instructive because the modified coefficient matrix $\hat{\mathbf{A}}$ determines whether the variables placed on the right hand side (z in this example) are arbitrary. When $\hat{\mathbf{A}}$, which will always be a square matrix, is nonsingular, $\hat{\mathbf{A}}\underline{\hat{\mathbf{x}}} = \underline{\hat{\mathbf{b}}}$ has a unique solution and the variables moved to the right hand side are indeed arbitrary.

Consider what happens when x is selected to be the arbitrary variable. The reduced 2 by 3 system with x as the arbitrary unknown becomes

$$\hat{\mathbf{A}}\underline{\hat{\mathbf{x}}} = \underline{\hat{\mathbf{b}}} \quad \text{where} \quad \hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \underline{\hat{\mathbf{b}}} = \begin{pmatrix} -x \\ -1 \end{pmatrix}, \quad \underline{\hat{\mathbf{x}}} = \begin{pmatrix} y \\ z \end{pmatrix}$$

and attempting to solve for a solution by Gauss-Jordan gives

$$\left(\hat{\mathbf{A}}|\underline{\hat{\mathbf{b}}}\right) = \left(\begin{array}{cc|c} 1 & 1 & -x \\ 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & x-1 \end{array}\right)$$

In this case, a unique solution for y and z in terms of x was not possible. This comes as no surprise because $\hat{\mathbf{A}}$ is clearly singular. This confirms what was already known from the previous case where z was arbitrary but x was constrained to be 1. In fact, the Gauss-Jordan solution above implies that x-1 = 0 for the equations to be consistent.

For larger systems with arbitrary unknowns, it is essential to check the modified coefficient matrix $\hat{\mathbf{A}}$ before continuing on with a Gauss-Jordan or backward substitution under the assumption that one or more specific variables can be arbitrary. The following system demonstrates this point.

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 - 2x_6 &= 3 \\
x_2 + x_4 - 3x_6 &= 0 \\
x_1 + x_3 - x_4 + 2x_5 + 4x_6 &= 4 \\
2x_1 - x_2 - x_3 - 4x_6 &= 1 \\
-x_1 + 3x_2 + 2x_3 + x_4 + 2x_6 &= 1 \\
x_2 + x_3 - 2x_4 - x_5 + 8x_6 &= 0
\end{aligned}$$

The echelon form of the augmented matrix is given below.

$$(\mathbf{A}|\underline{\mathbf{b}}) \sim \left(\begin{array}{cccccc|c}
1 & 1 & 1 & 1 & 1 & -2 & 3 \\
0 & 1 & 0 & 1 & 0 & -3 & 0 \\
0 & 0 & 1 & -3 & -1 & 11 & 0 \\
0 & 0 & 0 & 1 & -1 & -3 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

The original 6 by 6 system of equations has been converted to a new 5 by 6 system with a single arbitrary unknown. From the echelon form these equations are

$$\begin{array}{ccccccccccccc}
x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & - & 2x_6 & = & 3 \\
 & & x_2 & & & + & x_4 & & & - & 3x_6 & = & 0 \\
 & & & & x_3 & - & 3x_4 & - & x_5 & + & 11x_6 & = & 0 \\
 & & & & & & x_4 & - & x_5 & - & 3x_6 & = & -1 \\
 & & & & & & & & x_5 & & & = & 1
\end{array}$$

Clearly x_5 is not arbitrary. The modified matrix $\hat{\mathbf{A}}$ obtained from the columns of the echelon form with the x_5 column omitted is shown below. Quite obviously its singular, as expected, confirming that x_5 is not arbitrary.

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 & 11 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

However, there may be other non-arbitrary variables in addition to x_5 which are not as obvious. For example, suppose we proceed to solve the reduced 5 by 6 system using Gauss-Jordan with x_2 selected as the arbitrary unknown. Observe what happens.

$$\left(\hat{\mathbf{A}}|\underline{\hat{\mathbf{b}}}\right) = \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -2 & 3-x_2 \\ 0 & 0 & 1 & 0 & -3 & -x_2 \\ 0 & 1 & -3 & -1 & 11 & 0 \\ 0 & 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -2 & 3-x_2 \\ 0 & 1 & -3 & -1 & 11 & 0 \\ 0 & 0 & 1 & 0 & -3 & -x_2 \\ 0 & 0 & 1 & -1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -2 & 3-x_2 \\ 0 & 1 & -3 & -1 & 11 & 0 \\ 0 & 0 & 1 & 0 & -3 & -x_2 \\ 0 & 0 & 0 & -1 & 0 & -1+x_2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -2 & 3-x_2 \\ 0 & 1 & -3 & -1 & 11 & 0 \\ 0 & 0 & 1 & 0 & -3 & -x_2 \\ 0 & 0 & 0 & 1 & 0 & 1-x_2 \\ 0 & 0 & 0 & 0 & 0 & x_2 \end{array}\right)$$

The bottom row of zeros in the first 5 columns signifies that a solution for x_6 is impossible when x_2 is chosen to be arbitrary and the Gauss-Jordan method terminates without a solution. Furthermore, for consistency the last row implies that x_2 must be zero, further evidence it can't be arbitrary. The 4th row represents the equation

$$x_5 = 1 - x_2$$

which is consistent with the last row of the echelon form which states that $x_5 = 1$.

In problems of this type, the prudent thing to do is verify that the modified coefficient matrix $\hat{\mathbf{A}}$ is nonsingular before proceeding to find a solution. In the previous example, when x_2 was assumed to be arbitrary, $\hat{\mathbf{A}}$ became

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & -3 & -1 & 11 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

From MATLAB, its easy to verify that $\hat{\mathbf{A}}$ is singular and therefore x_2 should not be chosen as arbitrary.

```
A =

    1    1    1    1   -2
    0    0    1    0   -3
    0    1   -3   -1   11
    0    0    1   -1   -3
    0    0    0    1    0

EDU» det(A)

ans =

    0
```

The same approach applies when more than one variable is arbitrary. To illustrate, consider the system of equations

$$
\begin{array}{rcrcrcrcrcr}
x_1 & + & x_2 & + & x_3 & - & x_4 & - & 3x_5 & = & 3 \\
x_1 & - & x_2 & - & x_3 & - & 3x_4 & + & x_5 & = & -1 \\
 & & 2x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & 4 \\
x_1 & & & + & 2x_3 & & & - & 5x_5 & = & 1 \\
2x_1 & - & x_2 & + & x_3 & - & 3x_4 & - & 4x_5 & = & 0 \\
3x_1 & - & 2x_2 & & & - & 6x_4 & - & 3x_5 & = & -1
\end{array}
$$

The echelon form of the augmented matrix has three rows consisting entirely of zeros. The first three non-zero rows are

$$
\left(\begin{array}{ccccc|c}
1 & 1 & 1 & -1 & -3 & 3 \\
0 & 1 & 1 & 1 & -2 & 2 \\
0 & 0 & 1 & 1 & -2 & 0
\end{array}\right)
$$

Alternatively, the row reduced echelon form from MATLAB is shown below.

```
Ab =

     1     1     1    -1    -3     3
     1    -1    -1    -3     1    -1
     0     2     1     1    -2     4
     1     0     2     0    -5     1
     2    -1     1    -3    -4     0
     3    -2     0    -6    -3    -1

EDU» rref(Ab)

ans =

     1     0     0    -2    -1     1
     0     1     0     0     0     2
     0     0     1     1    -2     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
```

In the 3 by 5 system of equations that correspond to either echelon form, there must be 2 arbitrary unknowns. To check if say x_4 and x_5 can be arbitrary, we look at the modified coefficient matrix $\hat{\mathbf{A}}$ that results when the columns for x_4 and x_5 are removed from the first echelon form.

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad |\hat{\mathbf{A}}| = 1$$

Since $\hat{\mathbf{A}}$ is a nonsingular matrix, there is a unique solution to $\hat{\mathbf{A}}\underline{\hat{\mathbf{x}}} = \underline{\hat{\mathbf{b}}}$

$$\text{where} \quad \underline{\hat{\mathbf{x}}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \underline{\hat{\mathbf{b}}} = \begin{pmatrix} 3 + x_4 + 3x_5 \\ 2 - x_4 + 2x_5 \\ -x_4 + 2x_5 \end{pmatrix}$$

The solution is

$$\underline{\hat{\mathbf{x}}} = \left(\hat{\mathbf{A}}\right)^{-1} \underline{\hat{\mathbf{b}}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 + x_4 + 3x_5 \\ 2 - x_4 + 2x_5 \\ -x_4 + 2x_5 \end{pmatrix} = \begin{pmatrix} 1 + 2x_4 + x_5 \\ 2 \\ -x_4 + 2x_5 \end{pmatrix}$$

i.e. $\quad x_1 = 1 + 2x_4 + x_5, \quad x_2 = 2, \quad x_3 = -x_4 + 2x_5, \quad x_4 = \text{arbitrary}, \quad x_5 = \text{arbitrary}$

Since we know from the previous solution that x_2 is not arbitrary, it should come as no surprise that any 3 by 3 submatrix formed from the first five columns of either echelon matrix (minus the zero rows) is destined to be singular if it excludes the second column, the one corresponding to x_2. The resulting 3 by 3 matrices obtained from the first echelon form are given below and the reader should verify that they are all singular.

$$\begin{pmatrix} -1 & -1 & -3 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{pmatrix} \; x_1 \text{ and } x_2 \text{ columns removed} \qquad \begin{pmatrix} 1 & -1 & -3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \; x_2 \text{ and } x_3 \text{ columns removed}$$

$$\begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \; x_2 \text{ and } x_4 \text{ colums removed} \qquad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \; x_2 \text{ and } x_5 \text{ columns removed}$$

The row reduced echelon form is even more explicit as to why x_2 cannot be arbitrary. It is clear from this echelon form that $x_2 = 2$ and hence not arbitrary.

Furthermore, removing the x_2 column from the row reduced echelon form leaves the following matrix

$$\left(\begin{array}{cccc:c} 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & -2 & 0 \end{array}\right)$$

Regardless of which additional column is removed for the 2nd arbitrary variable, the resulting 3 by 3 modified coefficient matrix $\hat{\mathbf{A}}$ will be singular, confirming that x_2 cannot be one of the two arbitrary unknowns.

In summary, either echelon form of the original augmented matrix will reveal the existence of arbitrary unknowns. The original m by n system of m equations in n unknowns will be reduced to an m_1 by n system where $m_1 \leq m$ indicating that $m - m_1$ equations from the original system were redundant. If m_1 is less than n, the existence of $n - m_1$ arbitrary unknowns is assured. A particular subset of $n - m_1$ unknowns is arbitrary provided the m_1 by m_1 submatrix of the echelon matrix obtained by removing the columns corresponding to the $n - m_1$ unknowns is nonsingular.

The row reduced echelon form in MATLAB will identify the arbitrary variables directly, i.e. any row with a single 1 in the first n columns implies the variable associated with that column is not arbitrary. For example, in the following 4 by 6 row-reduced echelon form, x_1 and x_5 cannot be arbitrary unknowns.

$$\text{rref}(\mathbf{A}|\underline{\mathbf{b}}) = \left(\begin{array}{cccccc:c} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 & 3 & 4 & 1 \\ 0 & 0 & 1 & 2 & 7 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$