Section 4 Lagrange Interpolating Polynomials

In the previous sections we encountered two different ways of representing the unique nth order (or lower) polynomial required to pass through a given set of n+1 points. Yet another way of writing the polynomial, constrained in the same fashion, is presented here. It is referred to as Lagrange's form of the interpolating polynomial.

Once again, we assume the existence of a set of data points (x_i, y_i), $i = 0, 1, \ldots, n$ obtained from a function $f(x)$ so that $y_i = f(x_i)$, $i = 0, 1, \ldots, n$. A suitable function for interpolation $I(x)$ is expressible as

$$I(x) = \sum_{i=0}^{n} L_i(x) \cdot f(x_i) \quad (4.1)$$

$$= L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1) + \ldots + L_n(x) \cdot f(x_n) \quad (4.1a)$$

The functions $L_i(x)$, $i = 0, 1, \ldots, n$ are chosen to satisfy

$$L_i(x) = \begin{cases} 0 & x = x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \\ 1 & x = x_i \end{cases} \quad (4.2)$$

Before we actually define the $L_i(x)$ functions, let's be certain we understand the implications of Equations (4.1) and (4.2). The best way to accomplish this is simply to choose a value for "n" and write out the resulting equations. Suppose we have the four data points $[x_i, f(x_i)]$, $i = 0, 1, 2, 3$. From Equations (4.1) with $n = 3$, the interpolating function $I(x)$ becomes

$$I(x) = \sum_{i=0}^{3} L_i(x) \cdot f(x_i) \quad (4.3)$$

$$= L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1) + L_2(x) \cdot f(x_2) + L_3(x) \cdot f(x_3) \quad (4.3a)$$

and it remains to be shown that $I(x)$ is identical to $f(x)$ when x is any one of the four data points. Evaluating $I(x)$ at x_0, x_1, x_2 and x_3,

$$I(x_0) = L_0(x_0) \cdot f(x_0) + L_1(x_0) \cdot f(x_1) + L_2(x_0) \cdot f(x_2) + L_3(x_0) \cdot f(x_3) \quad (4.4)$$

$$I(x_1) = L_0(x_1) \cdot f(x_0) + L_1(x_1) \cdot f(x_1) + L_2(x_1) \cdot f(x_2) + L_3(x_1) \cdot f(x_3) \quad (4.4a)$$

$$I(x_2) \quad = \quad L_0(x_2)\cdot f(x_0) + L_1(x_2)\cdot f(x_1) + L_2(x_2)\cdot f(x_2) + L_3(x_2)\cdot f(x_3) \qquad (4.4b)$$

$$I(x_3) \quad = \quad L_0(x_3)\cdot f(x_0) + L_1(x_3)\cdot f(x_1) + L_2(x_3)\cdot f(x_2) + L_3(x_3)\cdot f(x_3) \qquad (4.4c)$$

According to Equation (4.2), $L_0(x_0) = 1$ and $L_1(x_0) = L_2(x_0) = L_3(x_0) = 0$. Equation (4.4) is simplified as shown below.

$$\begin{aligned} I(x_0) &= L_0(x_0)\cdot f(x_0) + L_1(x_0)\cdot f(x_1) + L_2(x_0)\cdot f(x_2) + L_3(x_0)\cdot f(x_3) \\ &= 1\cdot f(x_0) + 0\cdot f(x_1) + 0\cdot f(x_2) + 0\cdot f(x_3) \\ &= f(x_0) \end{aligned}$$

By the same reasoning, $I(x_1) = f(x_1)$, $I(x_2) = f(x_2)$, and $I(x_3) = f(x_3)$ which means the interpolating function $I(x)$ passes through the given set of data points.

The analytical form of $I(x)$ depends on the functions $L_i(x)$, $i = 0,1, \ldots, n$ that satisfy Equation (4.2). They are called Lagrange coefficient polynomials and are defined as follows:

$$L_i(x) \quad = \prod_{j=0,1,\ldots,i-1,i+1,\ldots,n} \left(\frac{x - x_j}{x_i - x_j} \right) \qquad i = 0, 1, 2, \ldots, n \qquad (4.5)$$

The symbol $\prod$ in Equation (4.5) is a symbolic notation for multiplication in the same way the symbol $\sum$ denotes summation of its arguments. To better understand Equation (4.5), $L_i(x)$ are expressed in a more explicit form. Given below are expressions for $L_1(x)$ through $L_3(x)$ when there are four data points ($n = 3$).

$$i = 0, \qquad L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \qquad (4.6)$$

$$i = 1, \qquad L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \qquad (4.6a)$$

$$i = 2, \qquad L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \qquad (4.6b)$$

$$i = 3, \qquad L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \qquad (4.6c)$$

Notice that each Lagrange coefficient polynomial in Equation (4.6) is a third order polynomial as a result of the x^3 term in the numerator. For the general case when there are n+1 data points, the Lagrange coefficient polynomials L_i(x) in Equations (4.1) are nth order polynomials and therefore so is the interpolating function $I(x)$. Henceforth we shall represent the interpolating polynomial $I(x)$ in Equation (4.1) by $f_n(x)$ and refer to it as the Lagrange interpolating polynomial.

The advantage of Lagrange interpolation in comparison with the standard polynomial form (Equation 2.1) or the Newton divided-difference representation (Equation 3.1) is its simplicity. That is, the Lagrange interpolating polynomial can be determined without need of solving a system of simultaneous equations or performing repetitive calculations as in the case of Newton interpolating polynomials where a table of divided differences is required. Owing to the manner in which the Lagrange coefficient polynomials are defined in Equation (4.5), the Lagrange interpolating polynomial is written by inspection of the data points using Equation (4.1).

We demonstrate the procedure in the following example.

Example 4.1

The measured voltage as a function of time across the terminals of a 5 ohm load with three different types of batteries, each with a nominal rating of 1.5 volts, is tabulated below.

t (hours)	v (volts)		
	Rechargeable Ni-Cd 1st Charge	Rechargeable Alkaline 1st Charge	Rechargeable Alkaline 2nd Charge
0	1.40	1.40	1.35
1	1.30	1.17	1.15
2	1.00	1.10	1.05
3	0.40	1.05	1.00
4	0.05	0.90	0.40

Table 4.1 Voltage and Elapsed Time for Rechargeable Batteries

The Lagrange interpolating polynomial for each type of battery is obtained directly from the table. For the Ni-Cd battery, it is

$$f_4(t) = \frac{(t-t_1)(t-t_2)(t-t_3)(t-t_4)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)(t_0-t_4)} \cdot v(t_0) + \frac{(t-t_0)(t-t_2)(t-t_3)(t-t_4)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)(t_1-t_4)} \cdot v(t_1)$$

$$+ \quad \frac{(t-t_0)(t-t_1)(t-t_3)(t-t_4)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)(t_2-t_4)} \cdot v(t_2) + \frac{(t-t_0)(t-t_1)(t-t_2)(t-t_4)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)(t_3-t_4)} \cdot v(t_3)$$

$$+ \quad \frac{(t-t_0)(t-t_1)(t-t_2)(t-t_3)}{(t_4-t_0)(t_4-t_1)(t_4-t_2)(t_4-t_3)} \cdot v(t_4) \qquad (4.7)$$

$$f_4(t) = \frac{(t-1)(t-2)(t-3)(t-4)}{(0-1)(0-2)(0-3)(0-4)} \cdot 1.40 + \frac{t(t-2)(t-3)(t-4)}{(1-0)(1-2)(1-3)(1-4)} \cdot 1.30$$

$$+ \frac{t(t-1)(t-3)(t-4)}{(2-0)(2-1)(2-3)(2-4)} \cdot 1.00 + \frac{t(t-1)(t-2)(t-4)}{(3-0)(3-1)(3-2)(3-4)} \cdot 0.40$$

$$+ \frac{t(t-1)(t-2)(t-3)}{(4-0)(4-1)(4-2)(4-3)} \cdot 0.05 \qquad (4.8)$$

$$f_4(t) = \tfrac{1.4}{24}(t-1)(t-2)(t-3)(t-4) - \tfrac{1.3}{6}t(t-2)(t-3)(t-4) + \tfrac{100}{4}t(t-1)(t-3)(t-4)$$

$$- \tfrac{0.40}{6}t(t-1)(t-2)(t-4) + \tfrac{0.05}{24}t(t-1)(t-2)(t-3) \qquad (4.9)$$

The same procedure is used to obtain the fourth order Lagrange interpolating polynomials for the two alkaline batteries. Figure 4.1 shows the data points and the fourth order interpolating polynomial for each battery.

It is generally advisable to begin with lower order polynomials for interpolation and then increase the order in a systematic fashion until the results are acceptable. In other words, a reduced subset of data points is chosen based on the expected range of the interpolant. Suppose we have n+1 data points measured from a function $f(x)$ where the range of x is from $x_{\min}$ to $x_{\max}$. Assuming interpolation over the entire interval is required with a single polynomial, we might choose a third or possibly fourth order polynomial (assuming n>4) based on a judicious choice of four or five points that include both extremes. Visual inspection of the polynomial and its relationship to the entire set of data points will determine if a higher order polynomial is required and which additional data point should be included.

When an additional data point is required, the higher order polynomial will generally exhibit more curvature than the previous one throughout the interpolation interval. For example, Figure 4.2 shows polynomials of order n =1, 2, 3, and 4 required to pass through data sets of two, three, four, and five points respectively

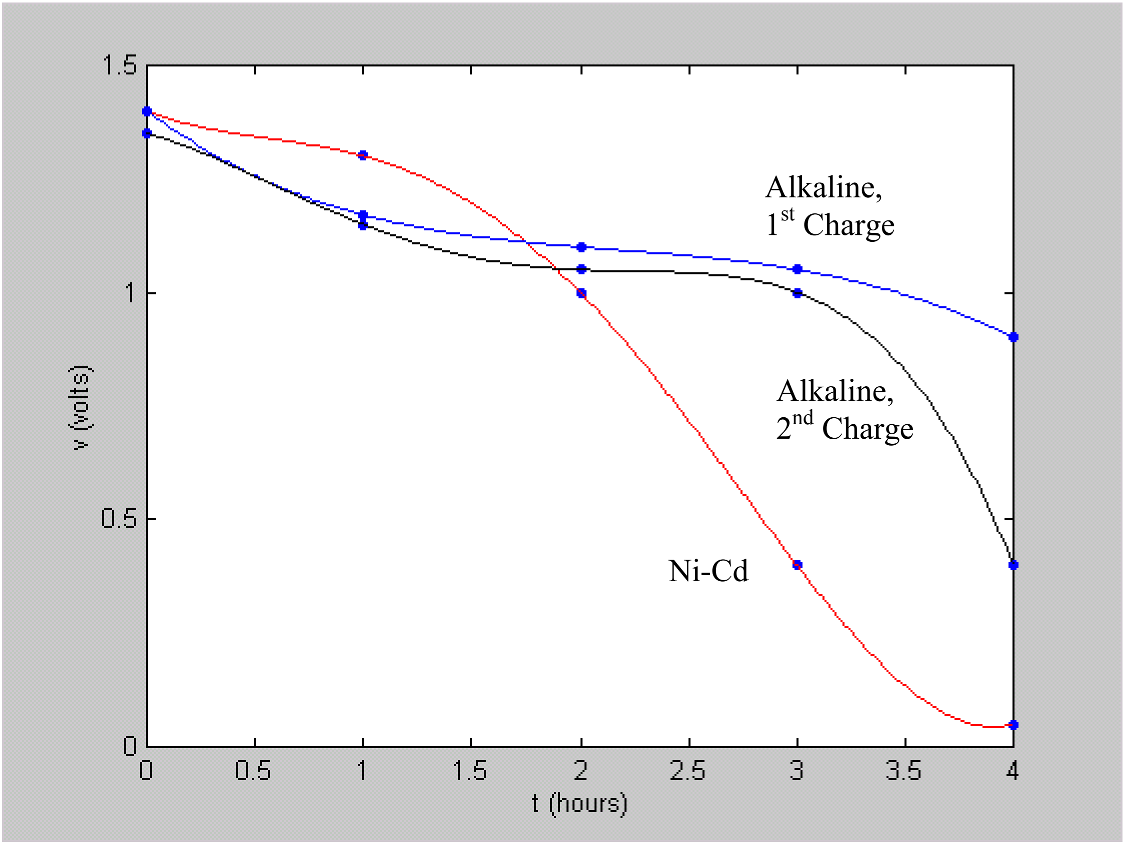


Figure 4.1 Fourth Order Interpolating Polynomials For Battery Data in Table 4.1

It's clear that the quadratic polynomial $f_2(x)$ exhibits more curvature than the linear function $f_1(x)$ which of course has none. This is possible due to the presence of the x^2 term in $f_2(x)$. Similarly $f_3(x)$ curves more than $f_2(x)$ due to the additional point $[x_3, f(x_3)]$ and the fourth order term x^4 in $f_4(x)$ accounts for the added curvature compared to $f_3(x)$ over the interval (0,4).

With an nth interpolating polynomial in standard form, Equation (2.1), or the Newton divided-difference form, Equation (3.1), there is only one high order term, i.e. a single term with x^n. This is in contrast to the Lagrange form of the interpolating polynomial, Equation (4.1), in which each term of the overall expression is an nth order polynomial. If the order of the interpolating polynomial is to be increased from n to n+1 by including an additional data point, each Lagrange coefficient polynomial in Equation (4.5) increases in order from n to n+1 as well. Consequently, the entire Lagrange interpolating polynomial must be recomputed.

Despite the fact interpolating polynomials in standard form contain a single high order term, an extra data point requires recalculation of all the coefficients a_i, $i = 0,1,2, \ldots, n$ in addition to the new coefficient a_{n+1}. The system of equations to be solved was considered in Section 3.2 and enumerated in matrix form in Equation (2.3). The Vandermonde matrix and the column vectors of coefficients and function values in Equation (2.3) are $(n+1)\times(n+1)$, $n\times 1$, and $n\times 1$, respectively.

The Newton divided-difference interpolating polynomial (of the three forms considered) minimizes the computational effort necessary to accommodate an additional data point. As we pointed out in Section 3.3, the Newton divided-difference interpolating polynomial $f_n(x)$ in Equation (3.1) is formulated in such a way that the coefficients b_i, $i = 1,2,3,\ldots,n$ do not change; only b_{n+1} is calculated using the expanded table of finite divided differences.

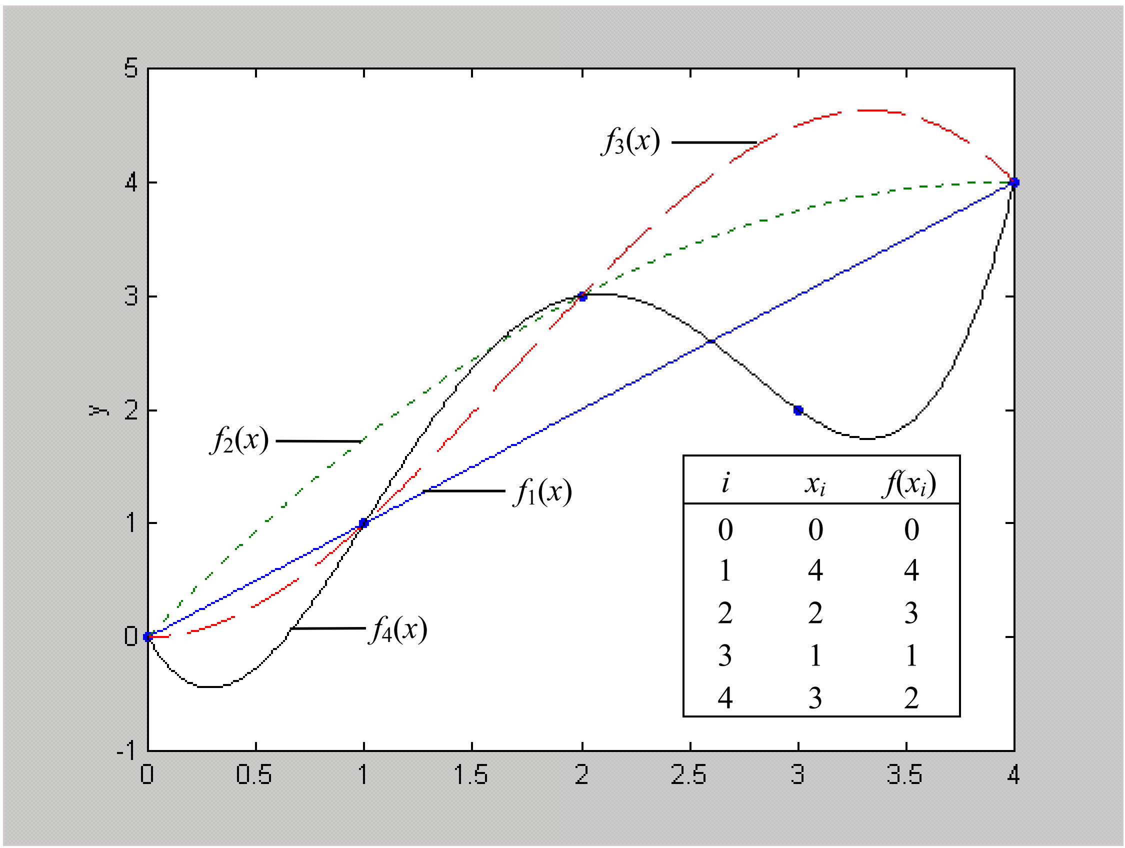


Figure 4.2 Increasing Curvature of a Polynomial as a Function of its Order

As with any interpolation method based on the use of approximation functions, there is an error term present which accounts for the difference between the true function value (usually unknown) and the interpolated value. Using an nth order Lagrange interpolating polynomial $f_n(x)$ to estimate values of $f(x)$, the error term $R_n(x)$ satisfies

$$f(x) = f_n(x) + R_n(x) \tag{4.10}$$

which is identical to Equation (3.31) in Section 3 when $f_n(x)$ was an nth order Newton divided-difference interpolating polynomial. Since $f_n(x)$ is unique for a given set of n+1 data points, much of the error analysis presented in Section 3 is applicable to Lagrange interpolating polynomials. Thus, an estimate of the error $R_n(x)$ is still the difference $f_{n+1}(x)$ - $f_n(x)$, where an additional data point is required to evaluate $f_{n+1}(x)$. The real advantage of Newton divided-difference interpolating polynomials is the reduced computational effort (compared with the Lagrange form) to obtain $f_{n+1}(x)$ when $f_n(x)$ is already computed.

Example 4.2

In Example 4.1, estimate the error incurred by the use of a fourth order Lagrange interpolating polynomial for approximating the voltage of a Ni-Cd battery after 1.5 hours. Assume an additional data point is available, namely t = 2.5 hours, v = 0.7 volts. From Equation (4.9),

$$\begin{aligned} f_4(1.5) &= \tfrac{1.4}{24}(1.5-1)(1.5-2)(1.5-3)(1.5-4) - \tfrac{1.3}{6}1.5(1.5-2)(1.5-3)(1.5-4) \\ &+ \tfrac{100}{4}1.5(1.5-1)(1.5-3)(1.5-4) - \tfrac{0.40}{6}1.5(1.5-1)(1.5-2)(1.5-4) \\ &+ \tfrac{0.05}{24}1.5(1.5-1)(1.5-2)(1.5-3) \\ f_4(1.5) &= 1.1965 \end{aligned}$$

Supplementing Table 4.1 with the new data point and determining the fifth order Lagrange interpolating polynomial yields,

$$\begin{aligned} f_5(t) = {} & \frac{(t-1)(t-2)(t-3)(t-4)(t-2.5)}{(0-1)(0-2)(0-3)(0-4)(0-2.5)}\cdot 1.40 + \frac{t(t-2)(t-3)(t-4)(t-2.5)}{(1-0)(1-2)(1-3)(1-4)(1-2.5)}\cdot 1.30 \\ & + \frac{t(t-1)(t-3)(t-4)(t-2.5)}{(2-0)(2-1)(2-3)(2-4)(2-2.5)}\cdot 1.00 + \frac{t(t-1)(t-2)(t-4)(t-2.5)}{(3-0)(3-1)(3-2)(3-4)(3-2.5)}\cdot 0.40 \\ & + \frac{t(t-1)(t-2)(t-3)(t-2.5)}{(4-0)(4-1)(4-2)(4-3)(4-2.5)}\cdot 0.05 + \frac{t(t-1)(t-2)(t-3)(t-4)}{(2.5-0)(2.5-1)(2.5-2)(2.5-3)(2.5-4)}0.70 \end{aligned} \tag{4.11}$$

$R_4(1.5)$ is the difference between the true (unknown) function value $f(1.5)$ and the fourth order interpolating polynomial estimate $f_4(1.5)$, i.e.

$$R_4(1.5) = f(1.5) - f_4(1.5)$$

We can estimate $R_4(1.5)$ using

$$\begin{aligned} R_4(1.5) &\approx f_5(1.5) - f_4(1.5) \\ &\approx 1.2148 - 1.1965 \\ &\approx 0.0184 \end{aligned}$$

Exercises

1. A simplified mathematical model of traffic flow is based on the assumption that at any point along a road, the velocity of a car depends only on the density of cars in the immediate vicinity. At low densities, drivers are free to drive at their desired speed, called the mean free speed. At high densities, vehicle speeds are lower, ultimately approaching zero at the maximum density corresponding to bumper-to-bumper traffic. This model is applicable on single lane roads or tunnels where car passing is not allowed.

   Measurements of average velocity (in mph) and traffic density (vehicles/mile) were taken in a tunnel to determine its velocity-density profile.

Density, ρ (vehicles/mile)	Velocity, u (mph)
20.5	39.8
31.2	30.9
50.4	22.7
86.4	13.1
120.9	6.2
160.0	4.3

Data for Problem 1

   a) Graph the data points.
   b) Find a third order Lagrange polynomial $u = f_3(\rho)$ suitable for interpolation over the entire range of densities. Plot the polynomial on the same graph as the data points.
   c) Calculate the true error $f(\rho) - f_3(\rho)$, where $u = f(\rho)$ is the true function, at the two remaining data points.
   d) Estimate the average speed of traffic when the mile and a half long tunnel has 75 cars uniformly spaced in it.
   e) Suppose the speed limit in the tunnel is 35 mph. Estimate the uniform traffic density throughout the tunnel at which traffic tends to flow at the speed limit.
   f) The traffic flow q (vehicles/hr) through the tunnel under steady-state conditions (uniform density and constant speed) is the product of traffic density, ρ (vehicles/mile) and vehicle speed, u (miles/hr). Generate a graph of q vs. ρ for the tunnel using the Lagrange polynomial from Part b). Traffic engineers refer to this as the Fundamental Diagram of Traffic Flow.

g) The maximum traffic flow is called the capacity of the road. Estimate the tunnel's capacity. What should the posted speed limit be to achieve this capacity?

2. The field strength of the earth's gravitational field g diminishes with distance h from the surface of the earth. The following table shows the field strength at several distances.

Distance, h (10^6 m)	Field Strength, g (N/kg)	Distance, h (10^6 m)	Field Strength, g (N/kg)
0	9.81	20	0.58
2.5	5.07	30	0.30
5	3.09	40	0.19
10	1.49	50	0.12

Data for Problem 2

a) Plot the data points.

b) An interpolating polynomial is required to estimate g from zero to 20 $\times 10^6$ m. Find the Lagrange interpolating polynomial $f_7(h)$ which passes through all the data points and the Lagrange interpolating polynomial $f_4(h)$ which passes through the first five data points. Plot both over the interval $0 \le h \le 20\times10^6$ m and comment on which is more suitable for interpolation.

c) Estimate the distance from earth where a 1 kg mass weighs 2 Newtons, i.e. where the gravitational field strength is 2 N/kg, using the two interpolating polynomials form Part b).

d) The true function relating g and h is

$$g = f(h) = \frac{Gm_E}{(R_E + h)^2}$$

where

where	$G = 6.7\times10^{-11}$ N·m²/kg²	(gravitational constant)
	$m_E = 6.0\times10^{24}$ kg	(mass of the earth)
	$R_E = 6.4\times10^6$ m	(radius of the earth)

Plot the true function $f(h)$ on the same graph as the data points and the interpolating polynomials $f_4(h)$ and $f_7(h)$ over the interval $0 \le h \le 50\times10^6$ m. Comment on the suitability of either interpolating polynomial over the entire interval $0 \le h \le 50\times10^6$ m.

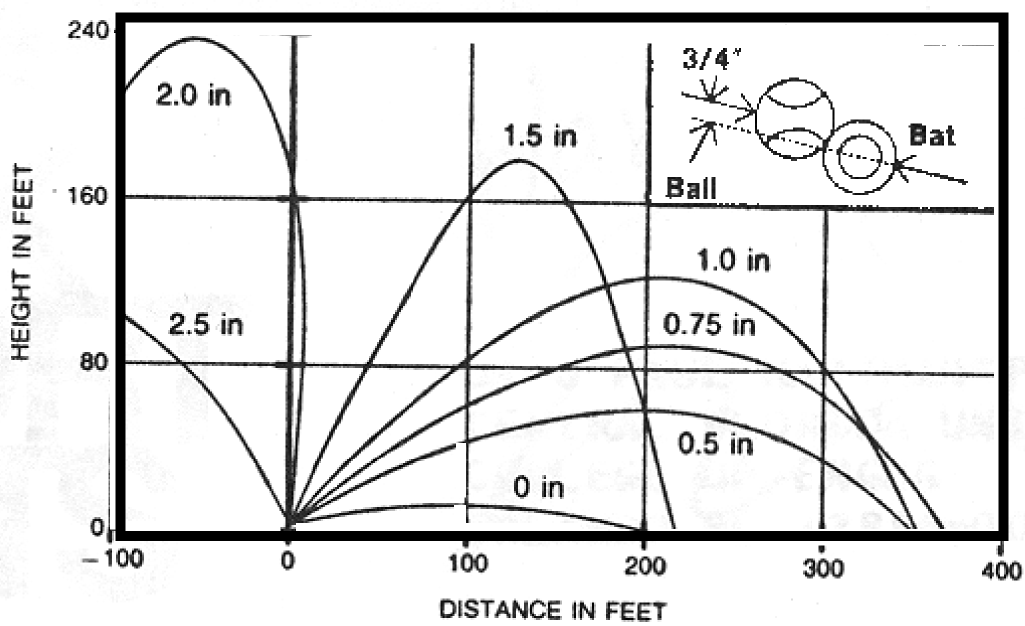
e) Calculate the true errors in the two estimates of g in Part c).

f) Try to find an interpolating polynomial that can be used to provide reasonably accurate estimates of g over the entire interval $0 \le h \le 50 \times 10^6$ m. Generate more data points from the true function $g = f(h)$ if you think this will help.

g) Use an appropriate Lagrange interpolating polynomial to estimate the distance from the earth's surface where you weigh one half of your weight on earth.

h) Calculate the error in the estimated distance in Part g).

3. The curves shown below are the trajectories of a batted baseball thrown at 85 mph with a bat speed of 70 mph directed upward by 10 degrees. The various curves correspond to the amount that the bat is swung under the ball. For example, the case when the underswing is ¾ inches is illustrated in the upper corner. (Source: Falk Sport Facts, F.W. Roth, 1991)



Graphs for Problem 3

a) Select one of the curves and generate a set of tightly spaced data points sampled from the curve. Represent the actual curve by a piecewise linear approximation through the data points. Graph the approximating function, which should appear smooth, as a dotted curve without showing the data points. Label it $H = f(D)$.

b) For the same curve selected in Part a) generate a reduced set of sampled data points (no more than 10) that captures the basic profile of the curve.

c) Try to fit a Lagrange interpolating polynomial $f_n(D)$ over the entire interval represented by the data points. Plot the piecewise function $f(D)$, the interpolating polynomial $f_n(D)$ and the data points on the same graph.

4. An electronics lab experiment was designed to compare the measured voltage-current relationship for a light emitting diode with calculated values based on the known nonlinear resistance characteristic of the diode. Results are given in the table below. (Source: Monaghan, Computers in Education, Jan-Mar 1998)

VOLTAGE VS. CURRENT FOR A LIGHT EMITTING DIODE			
LAB MEASUREMENTS		CALCULATED VALUES	
VOLTAGE (volts)	CURRENT (milliamperes)	VOLTAGE (volts)	CURRENT (milliamperes)
1.421	0.095	1.420	0.118
1.441	0.155	1.440	0.186
1.446	0.281	1.460	0.295
1.487	0.461	1.480	0.468
1.512	0.808	1.500	0.741
1.535	1.390	1.520	1.173
1.556	2.350	1.540	1.859
1.557	4.080	1.560	2.944
1.596	6.720	1.580	4.664

Table for Problem 4

a) Plot the measured and calculated current vs. voltage data points on separate graphs.

b) Find second, third and fourth order Lagrange interpolating polynomials for each set based on the following data points.

MEASURED:

ORDER	DATA POINTS
2	(1.421,0.095),(1.512, 0.808),(1.596,6.720)
3	(1.421,0.095),(1.487, 0.461),(1.535,1.390),(1.596,6.720)
4	(1.421,0.095),(1.446, 0.281),(1.512, 0.808), 1.556, 2.350),(1.596,6.720)

CALCULATED:

ORDER	DATA POINTS
2	(1.420,0.118), (1.500, 0.741), (1.580,4.664)

3 (1.420,0.118), (1.480, 0.468), (1.520,1.173), (1.580,4.664)

4 (1.420,0.118),(1.460, 0.295),(1.500, 0.741),(1.540,1.859), (1.580,4.664)

c) Plot the polynomials on the same graphs with the data points. Comment on the results.

d) Denote $i = f(v_i)$ as the function relating the actual measured current and voltage. Calculate the average error for each interpolating polynomial $f_n(v_i)$ i.e.

$$E_{ave} = \tfrac{1}{9}\sum_{i=1}^{9}\left[f(v_i) - f_n(v_i)\right], \qquad n = 2,3,4$$

e) Repeat Part d) with $i = f(v_i)$ as the function relating the actual calculated current and voltage.

5. The displacement diagram for a disk cam with reciprocating follower is shown in the figure.

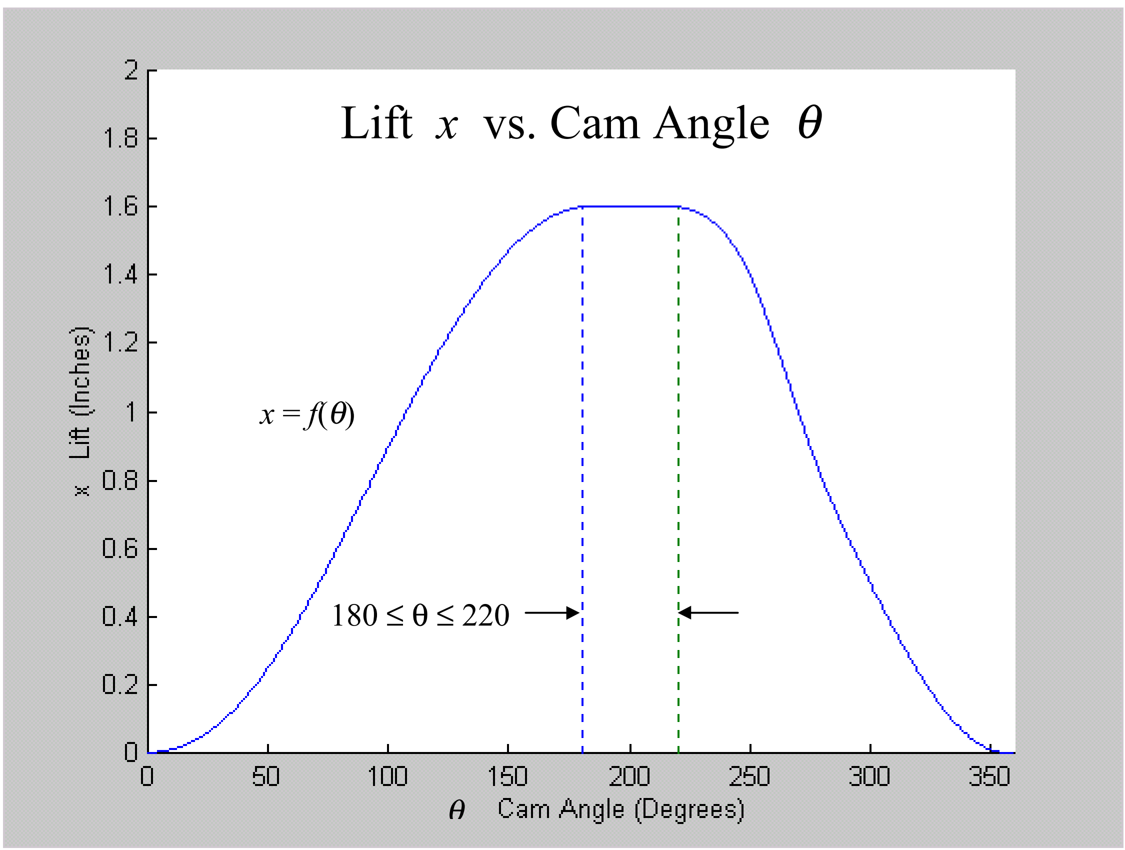


Figure for Problem 5

a) Generate two sets of data points (θ_i, x_i) one for cam angles θ_i between 0 and 180 degrees, the other for cam angles between 220 and 360 degrees. Choose the points so that Lagrange interpolating polynomials can be used to approximate both profiles. Plot both sets of data points on the same graph.

b) Find the lowest order Lagrange interpolating polynomials that describe both profiles reasonably well. (They do not have to be the same order)

c) Plot the polynomials from Part b) on the same graph as the data points.

d) Select several additional data points from both profiles and calculate the true error $f(\theta_i)-f_n(\theta_i)$, where n is the order of the polynomial, at each of those points.

6. A car buyer is financing the purchase of a $35,000 automobile. The bank has several loan packages to choose from. The choices are summarized below.

Loan Period n (months)	Annualized Interest rate i (%)	Monthly Payment A ($)
24	5.0	1535.50
36	5.5	1056.86
48	6.0	821.86
60	6.5	684.82
72	7.0	596.72

Table for Problem 6

a) Plot the data points (n_k, A_k), $k = 1,2,3,4,5$.

b) Find the 4th order Lagrange interpolating polynomial $A = f_4(n)$ through the data points and plot it on the same graph.

c) The bank is willing to write the loan for any number of months between 24 and 72. Estimate the monthly payment if the car buyer chooses a three and a half year loan.

d) The interest rate on the loan increases with the duration of the loan. Use the table to interpolate $i(42)$, the annualized per cent interest rate for a forty-two month loan.

e) The formula for calculating the monthly payment A given the amount financed P, the annual per cent interest rate i, and the loan duration n in months, is

$$A = f(P,i,n) = P\left[\frac{\frac{i}{1200}(1+\frac{i}{1200})^n}{(1+\frac{i}{1200})^n - 1}\right]$$

Find the true error in the estimated monthly payment $f_4(42)$ calculated in Part c), i.e. $f(35000, i(42), 42) - f_4(42)$.

f) The car buyer can afford monthly payments of $770. Use the graph $A = f_4(n)$ to estimate (to the nearest month) the loan duration requiring this amount of payment.

g) Find the true loan duration (to the nearest month) when the monthly payment is \$770. and compare the result to the answer in Part f). Hint: You will need to solve $A = f(P, i, n)$ for n by trial and error since i is unknown as well.

7. The estimated net lifetime benefits (in inflation adjusted 1993 dollars) for social security recipients born in different years is tabulated below. (Source: U.S. News and World Report, April 20, 1998)

Birth Year	Net Lifetime Benefits
1885	\$29,000
1900	\$51,000
1915	\$60,000
1930	\$14,000
1945	-\$18,000
1960	-\$33,000
1975	-\$32,000
1985	-\$31,000

Data for Problem 7

a) Graph the data points.

b) Find the seventh order Lagrange interpolating polynomial passing through the data points and plot it on the same graph.

c) Estimate the year of birth for a social security recipient to have net benefits of zero.

d) Estimate the maximum net lifetime social security benefits and the year of birth for recipients receiving the maximum.

8. Consider the 3 points from the function $f(x) = e^x$ in the table below.

i	x_i	$y_i = f(x_i)$
0	ln1	1
1	ln12	12
2	ln7	7

Table for Problem 8

a) Find the first order Lagrange polynomial $f_1(x)$ to be used for estimating $f(\ln 5)$.

b) Find the second order Lagrange polynomial $f_2(x)$ to be used for estimating $f(\ln 5)$.

c) Plot the given data points and both interpolating polynomials on the same graph.

d) Calculate the interpolation errors $R_i(\ln 5) = f(\ln 5) - f_i(x)$, $i = 1, 2$.

9. Railroad freight traffic measured in billions of ton-miles since 1965 is presented in tabular form. (Source: Assoc. of American Railroads)

Year	1965	1970	1975	1980	1985	1990	1995
Traffic	700	750	750	925	850	980	1275

Data Points for Problem 9

Estimate the year between 1975 and 1985 when the freight traffic was a maximum and the amount of traffic in that year.

10. The following data points were obtained from a function $y = f(x)$.

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	-10	-8	-6	-4	-2	0	2	4	6	8	10

a) Plot the points.

b) Find the lowest order Lagrange interpolating polynomial through the entire set of data points and plot it on the same graph.

c) Suppose the middle data point was incorrectly obtained as (5, y) where $y = 0.5$. Find the lowest order interpolating polynomial $f_n(x)$, $n \leq 10$, through the entire set of data points and plot it on the same graph. The interpolating polynomial may be expressed in standard form.

d) Find the second order Lagrange interpolating polynomial $f_2(x)$ through the end points and the new middle data point and plot it on the same graph.

e) Assuming the function $f(x)$ is linear, calculate the true error at $x = 0.4$ and $x = 4.75$ resulting from the use of both polynomials.

f) Repeat Parts c), d) and e) for $y = -1$ and 2.

g) Comment on the results.