Generic and Brand Advertising Strategies in a Dynamic Duopoly*

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Abstract

To increase the sales of their products through advertising, firms must integrate their brand advertising strategy for capturing market share from competitors and generic advertising strategy for increasing primary demand for the category. This paper examines whether, when, and how much brand advertising versus generic advertising should be done. Using differential game theory, optimal advertising decisions are obtained for a dynamic duopoly with symmetric or asymmetric competitors. We show how advertising depends on the cost and effectiveness of each type of advertising for each firm, the allocation of market expansion benefits, and the profit margins determined endogenously from price competition. We find that generic advertising is proportionally more important in the short term, and that there are free-riding effects leading to sub-optimal industry expenditure on generic advertising that worsen as firms become more symmetric. Due to free-riding by the weaker firm, its instantaneous profit and market share can actually be higher. The effectiveness of generic advertising and the allocation of its benefits, however, have little effect on the long-run market shares, which are determined by brand advertising effectiveness. Extensions of the model show that market potential saturation leads to a decline in generic advertising over time.

Keywords: *Advertising; Generic advertising*; *Differential games; Dynamic duopoly; Optimal control*

1. Introduction

From the relationship, "Product Sales = Category Sales × Market Share," it follows that marketing decisions can increase sales only by increasing category sales volume, i.e., primary demand[1], or by increasing market share. When the relevant marketing instrument is advertising, we define advertising whose effect is to increase category sales as "generic advertising," and advertising whose effect is to gain market share as "brand advertising." Operationally, generic advertising generates new sales by targeting beliefs about the product category while downplaying or oftentimes not mentioning the sponsoring brand. In contrast, brand advertising provides consumers with information about the brand's value proposition that differentiates it from its competitors, thereby encouraging consumers to choose the advertised brand over competing brands (Krishnamurthy 2000, 2001).

Allocating funds to generic advertising has an interesting consequence for the firm. Since generic advertising promotes the general qualities of the product category, it benefits all the firms in the category regardless of whether or not they paid for the advertising. Competing firms can benefit from the firm's contribution by free-riding, i.e., by not spending significant amounts of their own money on generic advertising (Krishnamurthy 2000). In this paper, we determine how much different firms should contribute towards generic advertising, and how firm and competitive factors affect their contributions.

Characterizing the optimal advertising policies in the presence of competitive effects is of great interest to researchers. However, the extant research on this topic is limited to either static models that incorporate generic and brand advertising (e.g., Krishnamurthy 2000) or dynamic models that do not explicitly consider generic advertising (e.g., Sorger 1989, Chintagunta and

[1] The terms "primary demand" and "category demand" are used interchangeably.

Vilcassim 1992, Erickson 1992). Thus, an important contribution of this paper is to examine dynamic competition in brand and generic advertising. We will examine the relevant literature in the next section.

The current research answers the following questions: (1) What should be a firm's generic and brand advertising budget, and how should it be dynamically allocated? (2) How do the nature of competition (symmetric or asymmetric) and other firm and market characteristics affect this allocation? Answering these questions contributes to the substantive literature on optimal advertising strategies in the presence of competition. We model market expansion by specifying a modified form of the Lanchester model of combat (Little 1979). We model the two types of sales growth in a dynamic duopoly – via market expansion (size of the pie) through generic advertising and market share allocation (slice of the pie) through brand advertising. Since both advertising and sales are time-varying, optimal control methods are used to analyze the situation (Sethi and Thompson 2000). Specifically, the Hamilton-Jacobi or "value function approach" is used to obtain explicit closed-loop Nash equilibrium solutions for the optimal advertising decisions (Villas-Boas 1999). We examine both symmetric and asymmetric competition.

The rest of the paper is organized as follows: In Sections 2 and 3, we review the existing literature. Section 4 presents the model. Section 5 deals with the analysis and the results for symmetric firms. Section 6 presents the results for asymmetric firms. Section 7 examines extensions of the basic model. Section 8 contains a discussion of the results. Finally, Section 9 concludes with a summary and directions for future research.

2. Literature Review

The effect of advertising on sales is an oft-researched topic (e.g., Bass and Parsons 1969). The bulk of this literature is devoted to brand advertising, justified by casual empirical evidence that suggests it to be the more common form of advertising. Thus, we first elaborate on the less known merits of generic advertising. We will then be in a better position to consider the relevant managerial decisions of budgeting and allocation of brand versus generic advertising.

Generic advertising increases primary demand by attracting new consumers, increasing per capita consumption of the product and lengthening the product life cycle (Friedman and Friedman 1976). Let us consider these cases in more detail:

a) New product categories: Generic advertising is particularly effective in the introductory stage of the product life cycle. Consumers are unaware of the product's uses and benefits and need to be informed and educated. For example, when P&G introduced Pampers diapers, it tried to enhance product acceptance by highlighting the advantages of using disposable diapers. Consider also advertisements by Sirius and XM, competitors in the nascent market for satellite radio. Douglas Wilsterman, Marketing VP of Sirius, says, "You've got to do a little bit of both [*viz. brand and generic ads*]. You can't just talk about yourself without people knowing what you represent in terms of a revolutionarily dynamic change" (Beardi 2001). Along similar lines, Steve Cook, XM's VP of Sales and Marketing, adds, "XM's ads will be about continuing to "grow the whole category pie," rather than competing with Sirius" (Boston and Halliday 2003). Other examples of generic advertising in new product categories, such as high-definition TV and the recordable DVD format, show advertisements promoting the advantages of these new standards without touting brand-specific features. Although the brand name may be mentioned, advertisements ask customers to compare the new technology with their existing technology without differentiating each firm's brand from competitors offering the same technology.

b) Increased penetration of mature products: Firms use generic advertising to market mature products by promoting new uses. Examples include Arm and Hammer's informing the public of new ways to use baking soda and Skippy showing consumers nontraditional ways to enjoy peanut butter. Other attempts to increase the sales of established products can be seen in De Beers' advertisement urging customers to buy diamonds for all occasions (Bates 2001), Dannon promoting the benefits of yogurt consumption, Norelco's "Gotcha!" campaign about the advantages of electric shavers over razors, and the Trojan condoms advertising campaign in the 1970s stressing the importance of family planning (Friedman and Friedman 1976).

c) Commodities: The critical role of generic advertising in the promotion of commodities can be traced to the 1950s when producers of tea and butter used generic advertising to compete with makers of coffee and margarine, respectively. Over the years, dairy producers have invested hundreds of millions of dollars in promoting the consumption of milk and other dairy products. These include the well-known "milk mustache" campaign by the California Milk Advisory Board, advertisements for eggs by the American Egg Board, and the "Pork: The Other White Meat" campaign. The advertisement for cotton, "The Fabric of Our Lives," for plastics by the American Plastics Council, and the California Raisin Advisory Board's promotion of raisins as a wholesome natural snack for children also come to mind in this regard. Other associations have used generic advertising to promote lamb, grapes, oranges, savings bank, life insurance, and the importance of having regular eye checkups (Friedman and Friedman 1976). When firms form an association to manage the generic advertising campaign, the basic rule is one of *co-opetition*, i.e., cooperate first and then compete, which suggests that firms can gain advantage by means of a judicious mixture of cooperation and competition (Brandenburger and Nalebuff 1997).

These examples show the range of uses of generic advertising. However, from these examples, no intuition emerges about when to emphasize generic advertising spending (initial or later stages of the product life cycle), under which circumstances to use it (market characteristics, e.g., high-tech, commodity, or other markets), or what effect competitive structure (symmetric or asymmetric competition) has on the decision. Consequently, a modeling approach is required. In this paper, we will broadly cover the issues of whether, when, and how much generic versus brand advertising should be undertaken.

We next examine the literature on generic and brand advertising.

3. Modeling Background

Researchers have studied competitive advertising strategies of firms using the theory of differential games (e.g., Deal 1979, Deal, Sethi, and Thompson 1979, Erickson 1985, Chintagunta and Vilcassim 1992, Erickson 1992, Chintagunta 1993, Espinosa and Mariel 2001, Prasad and Sethi 2004. Surveys of the literature are provided by Erickson (2003), Feichtinger, Hartl and Sethi (1994), Jorgensen (1982a), Jorgensen and Zaccour (2004), and Sethi (1977).) An example is the Lanchester model, given by

$$\begin{aligned}\dot{x}_1(t) &= \rho_1 u_1(t)(1-x_1(t)) - \rho_2 u_2(t)x_1(t), \quad x_1(0)=x_{10},\\ \dot{x}_2(t) &= \rho_2 u_2(t)(1-x_2(t)) - \rho_1 u_1(t)x_2(t), \quad x_2(0)=x_{20},\end{aligned} \tag{1}$$

where, for firm i, $i=1,2$, at time t, $x_i(t)$ is the market share, $u_i(t)$ is the brand advertising, and ρ_i is the effectiveness of advertising (Little 1979). Each firm uses brand advertising to capture market share from its rival. The model is a competitive extension of the Vidale-Wolfe (1957) model of advertising without the decay term in that model.

Sethi (1983) developed a variant of the Vidale-Wolfe model and used it to derive optimal advertising policies in a monopoly. Sorger (1989) extended the Sethi (1983) model to study brand advertising competition. The latter model is specified as

$$\begin{aligned}\dot{x}_1(t) &= \rho_1 u_1(t)\sqrt{1-x_1(t)} - \rho_2 u_2(t)\sqrt{x_1(t)}, \quad x_1(0) = x_{10},\\ \dot{x}_2(t) &= \rho_2 u_2(t)\sqrt{1-x_2(t)} - \rho_1 u_1(t)\sqrt{x_2(t)}, \quad x_2(0) = x_{20}.\end{aligned} \tag{2}$$

Sorger (1989) compares the model in (2) to other models of brand advertising, derives solutions for the optimal advertising expenditures, and also discusses various desirable properties of the model, notably the diminishing marginal returns to advertising and the fact that the structure can be made to resemble word-of-mouth and excess advertising models. Chintagunta and Jain (1995) find that this model fits the data from four product-markets (pharmaceutical, soft drink, beer, and detergent) well.

Krishnamurthy (2000, 2001) examines the relationship between generic and brand advertising but in a non-dynamic setting. The analysis suggests that if there is a dominant firm in the industry, the Nash equilibrium is for that firm to contribute everything and for the remaining firms to contribute nothing. Wrather and Yu (1979) also consider the static allocation of generic and brand advertising when there is a budget constraint.

Espinosa and Mariel (2001), Fruchter (1999), and Piga (1998) have proposed dynamic models of advertising competition with market expansion. However, in these models, both generic and brand advertising are modeled using a single advertising variable, so their separate effects on sales are not distinguished. Since sales responds differently to generic and brand advertising, their effects on sales should ideally be modeled separately.

A comparison of the various dynamic models of advertising competition in the literature is presented in Table 1.

<Insert Table 1 about here>

From this table, we see that generic and brand advertising decisions have not been simultaneously considered in a dynamic model. The present study specifically addresses this gap in the literature.

4. Model

Consider a dynamic duopoly with two firms labeled 1 and 2. We use the index $i=1,2$ to represent the two firms. We begin by listing the main notation:

$S_i(t)$	Sales of firm i at time t.
$u_i(t)$	Brand advertising of firm i at time t.
$a_i(t)$	Generic advertising of firm i at time t.
$p_i(t)$	Price charged by firm i at time t.
c_i	Advertising cost parameter for firm i.
ρ_i	Effectiveness of brand advertising of firm i.
k_i	Effectiveness of generic advertising of firm i.
θ_i	Allocation coefficient of firm i. We use $\theta_1+\theta_2=1$.
r_i	Discount rate for firm i.
$V_i(S_1(t),\ S_2(t))$	Value (or profit) function of firm i.

The market is one where advertising is the dominant marketing mix variable and other marketing mix decisions are less important or non-strategic. An example of a market with such features is the cola industry, dominated by Coke, Pepsi, and their "Cola Wars" (Chintagunta and Vilcassim 1992, Erickson 1992). We start by modeling the effect of generic advertising on category demand. The change in primary demand, $Q(t)$, is given by

$$\frac{dQ(t)}{dt}=\dot{Q}(t)=\dot{S}_1(t)+\dot{S}_2(t)=k_1a_1(t)+k_2a_2(t), \tag{3}$$

where $\dot{S}_i(t)$ is the rate of change of firm i's sales, $a_i(t)$ is the generic advertising of firm i, and k_i is the effectiveness of firm i's generic advertising for $i=1,2$. (When no confusion arises, we will drop the phrase $i=1,2$, as it applies whenever the index i is used. When an equation for firm i requires use of the argument relating to the other firm, we subscript this argument as $3-i$, since $i=1$ implies $3-i=2$ and $i=2$ implies $3-i=1$).

The increase in the category demand as a result of generic advertising is shared unequally. Let θ_i, the "allocation coefficient," denote the proportion of the sales increase that is transferred to firm i. The effect of generic advertising on firm i's sales, denoted $\dot{S}_{i,\,g}(t)$, is then

$$\dot{S}_{i,\,g}(t) = \theta_i\big(k_1 a_1(t) + k_2 a_2(t)\big). \tag{4}$$

To model the effect of brand advertising on sales, we modify the Sorger (1989) model given by (2) into a model of sales. The effect of brand advertising on firm i's sales, denoted $\dot{S}_{i,\,b}(t)$, is

$$\dot{S}_{i,\,b}(t) = \rho_i u_i(t)\sqrt{Q(t) - S_i(t)} - \rho_{3-i} u_{3-i}(t)\sqrt{S_i(t)}, \tag{5}$$

where $u_i(t)$ is the brand advertising decision of firm i and ρ_i is the effectiveness of that advertising. Hence, the brand advertising model is based on Sethi (1983) and other papers as discussed in the literature review. The total change in firm i's sales is $\dot{S}_i(t) = \dot{S}_{i,\,g}(t) + \dot{S}_{i,\,b}(t)$. Adding equations (4) and (5), the total effect of generic and brand advertising on firm i's sales rate is

$$\dot{S}_i(t) = \rho_i u_i(t)\sqrt{Q(t) - S_i(t)} - \rho_{3-i} u_{3-i}(t)\sqrt{S_i(t)} + \theta_i\big(k_1 a_1(t) + k_2 a_2(t)\big), \quad S_i(0) = S_{i0}, \tag{6}$$

where S_{i0} is the initial sales of firm i. Using $S_{3-i}(t) = Q(t) - S_i(t)$ yields

$$\dot{S}_i(t) = \rho_i u_i(t)\sqrt{S_{3-i}(t)} - \rho_{3-i} u_{3-i}(t)\sqrt{S_i(t)} + \theta_i\big(k_1 a_1(t) + k_2 a_2(t)\big), \quad S_i(0) = S_{i0}. \tag{7}$$

Equations (7) is intuitive in that the change in sales is the sum of sales gain due to its brand advertising, $\rho_i u_i(t)\sqrt{S_{3-i}(t)}$, minus the sales loss due to the rival's brand advertising, $\rho_{3-i}u_{3-i}(t)\sqrt{S_i(t)}$, plus the sales gain due to market expansion, $\theta_i(k_1a_1(t)+k_2a_2(t))$.

The control variables available to each firm are its generic and brand advertising decisions. Firm i's discounted profit maximization problem is

$$V_i(S_1,\ S_2)=\max_{u_i(t),\, a_i(t),\, p_i(t)}\int_0^\infty e^{-r_it}\big((1-b_ip_i(t)+d_ip_{3-i}(t))p_i(t)S_i(t)-C(u_i(t),\ a_i(t))\big)dt, \tag{8}$$

where r_i is its discount rate, $p_i(t)$ is the price charged, b_i and d_i are demand parameters, and $C(u_i(t),\ a_i(t))$ is the total advertising spending of firm i. The latter is specified as

$$C(u_i(t),\ a_i(t))=\frac{c_i}{2}\left(a_i(t)^2+u_i(t)^2\right). \tag{9}$$

As in most of the literature, the cost of advertising is assumed to be quadratic (e.g., Roberts and Samuelson 1988, Sorger 1989). Alternatively, one can use linear advertising costs and have advertising appear as a square root in the state equations.

The term $(1-b_ip_i(t)+d_ip_{3-i}(t))$ in the objective function multiplies the revenue $p_i(t)S_i(t)$ and is interpreted as the reduction in the margin $p_i(t)$ of firm i due to price competition.[2]

The discounted profit maximization problems of the two firms can now be rewritten as the differential game

[2] The term $(1-b_ip_i(t)+d_ip_{3-i}(t))$ can be alternatively, and equivalently, interpreted as reduction in sales due to price competition. In that case, the term $S_i(t)$ may be referred to as a state variable (which in equilibrium is sales of firm i times a constant). Note that constant marginal costs can be included without affecting the results.

$$
\begin{aligned}
V_1(S_1,\ S_2) &= \max_{u_1(t),\, a_1(t),\, p_1(t)} \int_0^\infty e^{-r_1 t}\Big((1-b_1p_1(t)+d_1p_2(t))\,p_1(t)S_1(t)-\frac{c_1}{2}\big(a_1(t)^2+u_1(t)^2\big)\Big)dt,\\
V_2(S_1,\ S_2) &= \max_{u_2(t),\, a_2(t),\, p_2(t)} \int_0^\infty e^{-r_2 t}\Big((1-b_2p_2(t)+d_2p_1(t))\,p_2(t)S_2(t)-\frac{c_2}{2}\big(a_2(t)^2+u_2(t)^2\big)\Big)dt,
\end{aligned} \tag{10}
$$

$$
\text{s.t.}\quad
\begin{aligned}
\dot{S}_1(t) &= \rho_1u_1(t)\sqrt{S_2(t)}-\rho_2u_2(t)\sqrt{S_1(t)}+\theta_1\big(k_1a_1(t)+k_2a_2(t)\big), \quad S_1(0)=S_{10},\\
\dot{S}_2(t) &= \rho_2u_2(t)\sqrt{S_1(t)}-\rho_1u_1(t)\sqrt{S_2(t)}+\theta_2\big(k_1a_1(t)+k_2a_2(t)\big), \quad S_2(0)=S_{20},
\end{aligned} \tag{11}
$$

where V_i is firm i's profit function, also known as the value function.

5. Analysis

The advertising differential game in (10-11) can be analyzed to yield either open-loop or closed-loop equilibria. Managers should find closed-loop strategies more useful since these strategies allow them to monitor the market and modify their advertising trajectories to respond to sudden changes in the marketplace (Erickson 1992). Chintagunta and Vilcassim (1992) and Erickson (1992) provide evidence that a closed-loop solution fits empirical data better than its open-loop counterpart. Therefore, we adopt the closed-loop solution concept. The closed-loop equilibrium is also the Markov Perfect equilibrium.

The optimal advertising policies are given in Proposition 1 (proof in Appendix):

Proposition 1: *The differential game* (10-11) *has a unique closed-loop Nash equilibrium solution for the two firms. For firm* i, $i=1,2$, *the optimal decisions are*:

a) Brand advertising: $$u_i^*(t)=\frac{\rho_i}{c_i}(\beta_i-\gamma_i)\sqrt{S_{3-i}(t)}, \tag{12}$$

b) Generic advertising: $$a_i^*(t)=\frac{k_i}{c_i}(\theta_i\beta_i+\theta_{3-i}\gamma_i), \tag{13}$$

c) Price: $$p_i^*(t)=\frac{d_i+2b_{3-i}}{4b_1b_2-d_1d_2}, \tag{14}$$

d) Value function: $$V_i(S_1,\ S_2)=\alpha_i+\beta_iS_i+\gamma_iS_{3-i}, \tag{15}$$

where α_1, α_2, β_1, β_2, γ_1, *and* γ_2 *solve the six simultaneous equations*

$$r_i\alpha_i - \frac{k_i^2}{2c_i}(\theta_i\beta_i + \theta_{3-i}\gamma_i)^2 - \frac{k_{3-i}^2}{c_{3-i}}(\theta_1\beta_1 + \theta_2\gamma_1)(\theta_2\beta_2 + \theta_1\gamma_2) = 0, \quad i = 1,2, \tag{16a}$$

$$r_i\beta_i - m_i + \frac{\rho_{3-i}^2}{c_{3-i}}(\beta_1 - \gamma_1)(\beta_2 - \gamma_2) = 0, \quad i = 1,2, \tag{16b}$$

$$r_i\gamma_i - \frac{\rho_i^2}{2c_i}(\beta_i - \gamma_i)^2 = 0, \quad i = 1,2. \tag{16c}$$

In view of (14), we replace the equilibrium margin $p_i^*(t)(1 - b_i p_i^*(t) + d_i p_{3-i}^*(t))$ in the objective function, which is a constant $b_i(\frac{d_i + 2b_{3-i}}{4b_1b_2 - d_1d_2})^2$, denoting it by m_i henceforth.

From Proposition 1(a), each firm's brand advertising is increasing in its competitor's sales. This follows from the specification of the brand advertising dynamics in (11) and is consistent with Sorger (1989). Proposition 1(b) and 1(c) suggest that the solutions for the optimal generic advertising and prices are stationary. This finding of stationary solutions is consistent with the analysis of an infinite horizon setting (Horsky and Mate 1988, Villas-Boas 1999).

We first consider the analysis for symmetric firms. In this case, $\alpha_i = \alpha$, $\beta_i = \beta$, $\gamma_i = \gamma$, $k_i = k$, $\rho_i = \rho$, $m_i = m$, $c_i = c$, $r_i = r$, and $\theta_i = \frac{1}{2}$ for $i = 1,2.$ Corollary 1 presents the solutions for symmetric firms (proof in Appendix).

Corollary 1 *(symmetric firms)*: *For the solution to the differential game* (10-11) *obtained in Proposition* 1, *the explicit solutions for the parameters* α, β, *and* γ *are given by*

$$\alpha = \frac{k^2\left(c^2r^4 - 3cmr^2\rho^2 + 18m^2\rho^4 + (6m\rho^2 - cr^2)\sqrt{cr^2(cr^2 + 6m\rho^2)}\right)}{108cr^3\rho^4}, \tag{17a}$$

$$\beta = \frac{3m\rho^2 - 2cr^2 + 2\sqrt{cr^2(cr^2 + 6m\rho^2)}}{9r\rho^2}, \tag{17b}$$

$$\gamma = \frac{3m\rho^2 + cr^2 - \sqrt{cr^2(cr^2 + 6m\rho^2)}}{9r\rho^2}. \tag{17c}$$

The comparative statics for the variables on the model parameters are presented in Table 2 (proof in Technical Appendix). A discussion of the comparative statics is in the next section.

<Insert Table 2 about here>

In the next section, we examine the solution given in Proposition 1 to the differential game (10-11) for asymmetric firms.

6. Asymmetric Firms

For asymmetric firms, explicit solutions for the set of simultaneous equations in (16a-c) are presented in the Appendix. The comparative statics for the parameters on the variables of interest are presented in Table 3 (proof in Technical Appendix). Note that α_i, β_i, and γ_i are functions of the model parameters only and not of time.

<Insert Table 3 about here>

One can see a few ambiguous effects. However, for most cases, the results are clear. The optimal generic advertising of a firm increases with an increase in the effectiveness of its generic advertising and the proportion of increase in its sales as a result of generic advertising. For the competitive effects, an increase in the rival's advertising cost parameter and increase in the effectiveness of the rival's generic advertising increase the firm's profit. An increase in one firm's brand advertising due to an increase in its effectiveness results in a decrease in the rival's brand advertising. The increase of one firm's profit with the effectiveness of the rival's generic advertising can be seen as evidence of the free-riding of generic advertising expenditures – the greater the effort exerted by the rival in developing better generic advertising copy, the better off the other firm is likely to be. The parameter for effectiveness of generic advertising and the proportion of sales increase due to generic advertising do not have an effect on the optimal brand advertising policies.

From the optimal advertising decisions, we can compute the long-run equilibrium market shares of the two firms. These are listed in the following proposition (proof in Appendix):

Proposition 2: *For the differential game given by equations* (10-11)*, the long-run market shares of the two firms are*

$$\left(\bar{x}_1 = \frac{\frac{\rho_1^2}{c_1}(\beta_1-\gamma_1)}{\frac{\rho_1^2}{c_1}(\beta_1-\gamma_1)+\frac{\rho_2^2}{c_2}(\beta_2-\gamma_2)}, \quad \bar{x}_2 = \frac{\frac{\rho_2^2}{c_2}(\beta_2-\gamma_2)}{\frac{\rho_1^2}{c_1}(\beta_1-\gamma_1)+\frac{\rho_2^2}{c_2}(\beta_2-\gamma_2)} \right), \tag{18}$$

where β_1*,* β_2*,* γ_1 *and* γ_2*, are obtained from Proposition* 1.

Although the long-run equilibrium market share in (18) is constant through time, its level is determined by the dynamic effects in the analysis. The form of the long-run equilibrium market shares in equation (18) resembles the $us/(us+them)$ market attraction models in the marketing literature. The market share attraction model given by (18) takes into account the effectiveness of brand advertising, the advertising cost parameter, and the other model parameters as measures of a firm's attractiveness. The equilibrium market share of the firm increases with that firm's brand advertising effectiveness and the rival's advertising cost parameter, and decreases with its advertising cost parameter and the effectiveness of the rival's brand advertising. Therefore, improving the effectiveness of brand advertising by developing better advertising copy has not only a short-term effect on the firm's sales, but also a long-run impact on the market share of the firm.

Note that the long-run market shares are unaffected by the parameter θ_i, $i=1,2,$ which captures the proportion of the marginal increase in market size obtained by firm i. This is because the market share calculation is driven primarily by the share of the existing market size and not by the share of the marginal increase (except during the introductory phase when the established market is quite small). The latter is determined by the brand characteristics and brand advertising.

In the next section, we turn our attention to the free-riding of generic advertising.

6.1. Free-riding

We consider the case of a pure monopoly to compare against generic advertising investment under competition. The results are summarized in Proposition 3 (proof in Appendix):

Proposition 3: *For the differential game given by* (10-11)*, if both brands are owned by the same firm; then*: *(a) Total generic advertising is higher than in the competitive case. (b) The optimal brand advertising is zero for one of the two brands, viz. the less profitable brand.*

A firm's coordinated decision making with regard to the two types of advertising is an example of category management for different brands in the firm's product line (Zenor 1994). The increase in generic advertising relative to the competitive case follows from the fact that there is no free-riding. Therefore, all the gains from generic advertising accrue to the monopolist.

Next, we use numerical analysis to determine the impact of free-riding on the profitability of the two firms. Let us define the "stronger" firm as the one with more favorable model parameters, i.e., higher effectiveness of generic and brand advertising, higher gross margin, etc., than the "weaker" firm.

Observations: *(a) The stronger firm should tolerate free-riding of generic advertising more than the weaker firm. (b) Free-riding impacts the stronger firm's profitability only in the short run. The stronger firm may obtain less profit in the initial periods than the weaker firm. However, in* *a de facto monopoly, the stronger firm is always more profitable. (c) As the degree of asymmetry increases, the sub-optimality of industry generic advertising decreases.*

For a wide range of parameter values, the weaker firm initially has higher profit than the stronger firm. This is because the weaker firm benefits from the stronger firm's generic advertising spending, while it takes some time for the stronger firm to recoup its investment. However, in the long run, the stronger firm still wins out, implying that free-riding does not offer a long-term market share and profit advantage to the weaker firm. This is referred to as the "big pig, little pig" game in game theory, wherein the "little pig" (weaker firm) profits from the efforts of the "big pig" (stronger firm). Commercials by IBM and De Beers and Campbell Soup's advertisements about the benefits of soup consumption and that soup can be used in casseroles and other dinner dishes illustrate this phenomenon.

When the asymmetry between the firms increases, there is a greater difference between their generic advertising contributions. Interestingly, the weaker firm's investment in generic advertising never goes to zero. Therefore, to use Krishnamurthy's (2000) terminology, there is "cheap-riding", but no "free-riding." In Krishnamurthy's model, free-riding can occur, possibly because future benefits are not considered.

As the degree of asymmetry grows, the sub-optimality of total industry generic advertising decreases. In other words, the free-riding problem is most serious when the firms are identical. In contrast, when one firm is a de facto monopolist, it spends on generic advertising near the optimal industry level. This explains the large generic advertising expenditures of firms such as Campbell Soup and De Beers.

7. Extensions

We consider some extensions of the basic model to determine the robustness of the results. First, we incorporate the idea of a market potential to provide an upper bound for the market demand. In another extension, we relax the assumption that brand advertising is done solely for offensive reasons. We also examine an endogenous specification of θ_i, the allocation coefficient.

7.1. Market Potential

The sales model in (11) is modified by introducing a market potential $\bar{Q}$ as follows:

$$\dot{S}_i(t) = \rho_i u_i(t)\sqrt{S_{3-i}(t)} - \rho_{3-i}u_{3-i}(t)\sqrt{S_i(t)} + \theta_i\big(k_1 a_1(t) + k_2 a_2(t)\big)\sqrt{\bar{Q} - S_1(t) - S_2(t)},\ S_i(0) = S_{i0}. \quad (19)$$

The square root formulation from the original model is retained, with $\sqrt{\bar{Q} - S_1(t) - S_2(t)}$ reflecting the fact that generic advertising informs the fraction of the market that is uninformed.

The profit maximization problems of the two firms are given, as before, by equation (10), subject to the state equations in (19). The analysis is in the Technical Appendix. The optimal advertising decisions are given by

$$u_i^*(t) = \frac{\rho_i}{c_i}(\beta_i - \gamma_i)\sqrt{S_{3-i}(t)}, \quad i = 1, 2, \quad (20)$$

$$a_i^*(t) = \frac{k_i}{c_i}(\theta_i\beta_i + \theta_{3-i}\gamma_i)\sqrt{\bar{Q} - S_1(t) - S_2(t)}, \quad i = 1, 2, \quad (21)$$

where the unknowns β_1, β_2, γ_1 and γ_2, are the solutions of the simultaneous equations:

$$r_i\beta_i = m_i - \frac{k_i^2}{2c_i}(\theta_i\beta_i + \theta_{3-i}\gamma_i)^2 - \frac{\rho_{3-i}^2}{c_{3-i}}(\beta_1 - \gamma_1)(\beta_2 - \gamma_2) - \frac{k_{3-i}^2}{c_{3-i}}(\theta_1\beta_1 + \theta_2\gamma_1)(\theta_2\beta_2 + \theta_1\gamma_2), \quad (22a)$$

$$r_i\gamma_i = -\frac{k_i^2}{2c_i}(\theta_i\beta_i + \theta_{3-i}\gamma_i)^2 + \frac{\rho_i^2}{2c_i}(\beta_i - \gamma_i)^2 - \frac{k_{3-i}^2}{c_{3-i}}(\theta_1\beta_1 + \theta_2\gamma_1)(\theta_2\beta_2 + \theta_1\gamma_2). \quad (22b)$$

From the optimal generic advertising decision in (21), one can see that the generic advertising is significantly greater when only a small fraction of the market potential has been realized than when most of the potential has been realized. This is because, if most of the market is untapped, the two firms have greater incentive to expand the market, and these investments decrease over time as the market becomes saturated. As more of the market potential is tapped, the two firms decrease their generic advertising, corresponding to an increase in brand advertising. In the extreme case, when the market is completely saturated, generic advertising is zero, and all resources are invested in brand advertising.

7.2. Other Extensions

To model the fact that brand advertising can be used to retain market share (defensive advertising), assume that a proportion ω_i of firm i's brand advertising is spent purely for offensive reasons and $1-\omega_i$ is spent for defensive reasons. Moreover, let θ_i, the allocation coefficient, be endogenized in terms of the sales levels of the two firms as $\theta_i = S_i^{\delta}/(S_1^{\delta}+S_2^{\delta})$, with $\delta<1$ capturing diminishing returns. In other words, the firm with greater sales obtains a greater fraction of the gain from market expansion.

The profit maximization problems of the two firms are given, as before, by equation (10), subject to

$$\dot{S}_i(t) = \rho_i u_i(t)\left(\omega_i\sqrt{S_{3-i}(t)} + (1-\omega_i)\sqrt{S_i(t)}\right) - \rho_{3-i}u_{3-i}(t)\left(\omega_{3-i}\sqrt{S_i(t)} + (1-\omega_{3-i})\sqrt{S_{3-i}(t)}\right) + \frac{S_i(t)^{\delta}}{S_1(t)^{\delta}+S_2(t)^{\delta}}\left(k_1a_1(t)+k_2a_2(t)\right),\ \ S_i(0)=S_{i0},\ \ i=1,2. \tag{23}$$

The analysis is presented in the Technical Appendix, and only the results from the simulation are discussed here. First, consider the effect of endogenizing θ_i in terms of the sales of the two firms. From the numerical results, we find that as δ increases from 0 to 1 in the symmetric case, i.e., when θ_i changes from 0.5 to the market shares of the two firms, the stronger firm allocates a greater chunk of its advertising budget toward generic advertising. This is because it gains proportionately more from generic advertising, since the value of θ_i for the stronger firm is always greater than 0.5. Consequently, the brand advertising of the stronger firm decreases as δ increases. The weaker firm's strategies, on the other hand, are not greatly affected by this change in δ because the value of θ_i for that firm is relatively low, so a greater fraction of its sales comes as a result of its high brand advertising outlay.

Moving to the effect of defensive advertising on the optimal advertising decisions, the simulation reveals that as ω decreases from 1 to 0, i.e., when more of the firm's brand advertising budget is expended for defensive reasons (retaining its customers), both firms increase their brand advertising significantly, with the stronger firm investing more because it has a greater share to defend. Corresponding to this increase in the brand advertising, the two firms decrease their generic advertising outlays. In the absence of defensive advertising, as in Section 4, the weaker firm would invest more in brand advertising since it has more to gain from its rival.

8. Discussion

We now discuss some salient features of the model and analysis, such as the need for integrated decision making with respect to generic and brand advertising, the tradeoff between generic and brand advertising, and the impact of free-riding.

Integrated Advertising Decisions: Coordinated decision making with regard to generic and brand advertising and price is an example of Integrated Marketing Communications (Naik and Raman 2003). In particular, the analysis in the paper suggests that the generic advertising strategy of a firm must be determined together with its brand advertising strategy to prevent sub-optimal advertising.

This conclusion is important given the literature review in Table 1, which shows that differential game models of advertising competition have not modeled generic advertising as a decision, but have recognized the need to model market expansion due to advertising. The assumption has been that there is some sort of leakage category expansion effect of brand advertising. However, it is impossible to say from these models how much a firm should invest in expanding the market versus winning market share due to the single control. As Krishnamurthy (2000) and others have discussed, it is an institutional feature of industries that firms decide how much to invest in market expansion programs. They do not rely on a leakage of their brand advertising efforts. It is also an institutional feature that advertising is done with specific objectives in mind. One can design advertising campaigns for whether one wants category expansion or market share. Advertising done with the purpose of expanding the category is termed generic advertising. We provide several examples of generic advertising in Section 2 of the paper. Having separate decision variables for brand and generic advertising is an important innovation in this paper. Although there are static game theory models that have examined the decisions separately, this paper provides a differential game that includes generic advertising as a theoretical contribution for the first time.

Budgeting and Allocation of Advertising: Propositions 1 and 2 completely describe the budget and allocation of advertising for the two firms in terms of the nature of competition and firm and market characteristics, thereby answering the research questions posed in the introduction.

One can get some insights into the allocation of the budget (into generic and brand advertising) by solving different problems that have the same optimal budget and different parameters, without imposing a budget constraint. In this approach, we note down the i^{th} firm's net present value of the total advertising budget, $\int_0^T e^{-r_i t} \frac{c_i}{2}(a_i^2 + u_i^2)dt,$ for a given set of parameter values. We then change the parameter values by trial and error in such a manner that the optimal budget remains unchanged. For example, we can change ρ_1, the first firm's brand advertising effectiveness, and find out the value of another parameter (e.g., k_1, the first firm's generic advertising effectiveness) that will result in the same NPV of the total advertising budget. Although the advertising budget remains the same, the allocation will be different from that obtained earlier.

Figure 1 illustrates this tradeoff between generic and brand advertising for the two cases ($\rho_1 = 0.6,\ k_1 = 0.6$ and $\rho_1 = 0.5,\ k_1 = 0.93$).

<Insert Figure 1 about here>

As can be seen from the plot, the optimal advertising trajectories for the two cases are different. As the parameter values change from $\rho_1 = 0.6,\ k_1 = 0.6$ to $\rho_1 = 0.5,\ k_1 = 0.93$, i.e., as the firm's brand advertising effectiveness decreases and the generic advertising effectiveness increases, the optimal brand advertising of the firm decreases, corresponding to an increase in its generic advertising. Given the same advertising budget, the decrease of one type of advertising corresponding to an increase in the other type is a measure of the tradeoff between the two types of advertising.

Expressing the generic and brand advertising expenditures as the share of an advertising dollar offers another measure of tradeoff between the investments in the two types of advertising. The percentage of generic advertising over time can be plotted for different values of the effectiveness of the two types of advertising (Figure 2). We use the same parameter values from the earlier analysis, i.e., $\rho_1 = 0.6,\ k_1 = 0.6$ and $\rho_1 = 0.5,\ k_1 = 0.93$. Figure 2 represents the split of a marginal dollar between generic and brand advertising at any time.

<Insert Figure 2 about here>

If the effectiveness of generic advertising is greater than that of brand advertising ($k > \rho$), the proportion of the budget invested in generic advertising is initially high, but decreases over time. If, on the other hand, both types of advertising have the same effectiveness ($k = \rho$), then we find that the proportion of the advertising budget invested in generic advertising is lower than in the previous case, but it increases initially before decreasing in later periods. Qualitatively, in all cases, generic advertising should be given relatively more importance in the initial periods (compared to later periods), and brand advertising should be given relatively more importance in the later periods (compared to initial periods).

Effect of Free-riding: Analysis shows that total generic advertising is higher if both brands are owned by the same firm than in the case of competition. The intuition behind this result is the commonly seen free-riding effect in public goods activities. Each firm must divide its gains with competitors; hence the marginal benefit from investing in generic advertising is diminished. As a result, industry growth is less than its cooperative-optimal. The implication for the industry is that it would be better to coordinate the activities of the firms. This explains why industry associations undertake the task of doing collective advertising for the industry. Examples such as the milk and plastics industry were discussed in Section 2 of the paper.

Under joint ownership, one would expect that there would be no brand advertising when the brands are jointly owned and that advertising under competition can be considered as a prisoner's dilemma. However, we find that in the case of joint ownership, optimal brand advertising is not zero for both brands but only for one brand. The intuition is that one brand is more profitable to the firm than the other, so the firm would like to drive demand towards that brand through brand advertising.

9. Conclusions

To increase the sales of its brand, a firm can use generic advertising to expand the entire market or brand advertising to win market share. The benefits of generic advertising are conferred to all firms regardless of who contributed. As a result, the generic advertising strategy of a firm must be integrated with its brand advertising strategy, necessitating a thorough understanding of the relationship between the two. However, the market expansion role of advertising has been under-studied relative to its share expansion role.

This paper explicitly considers market expansion and market share effects. We derive the closed-loop Nash equilibrium strategies for a dynamic duopoly where firms make decisions on generic and brand advertising. Explicit solutions are obtained for symmetric and asymmetric competitors. The effects of the model parameters on the optimal advertising policies and profits are found. A general conclusion is that generic and brand advertising must be properly coordinated, and neglecting one of the two will lead to sub-optimal allocation of the advertising budget. We also examined free-riding in generic advertising and its effect on the long-run profitability of the two firms, and find that although there is free-riding, the stronger firm is better off tolerating this free-riding since this does not affect its long-term profitability greatly.

Three extensions to the basic model were examined. The first deals with the inclusion of a market potential, the second with generalizing the allocation of gains from generic advertising, and the third with brand advertising being used for defensive reasons. Analyses of these extensions provide evidence of the robustness of the basic model.

Some limitations of the current study should also be acknowledged. Firstly, we do not micro-model the consumer purchase decision or disaggregate customers into separate segments, such as a loyal segment, or consider how brand preferences may be permanently altered through trial due to learning (Villas-Boas 2004), creating hysteresis effects. Secondly, unlike in Liu, Putler, and Weinberg (2004), product quality is exogenous and is not affected by competition. Finally, threshold effects in advertising response have been posited on and off (e.g., Vakratsas et al. 2004) but are not considered in the present model, which assumes a concave response.

Future research should consider including these effects and other alternative or more general specifications, including models with interaction between generic and brand advertising, and the use of contracts to achieve the cooperative-optimal generic advertising. One may also extend the model to study advertising competition in an oligopoly (Erickson 1995, Teng and Thompson 1984, Villas-Boas 1993). Finally, the comparative statics results can be tested empirically.

Appendix

Proof of Proposition 1

The Hamilton-Jacobi-Bellman (HJB) equation for firm i, $i=1,2$, is given by

$$r_i V_i = \max_{u_i,\, a_i,\, p_i} \begin{cases} p_i(1-b_i p_i + d_i p_{3-i})S_i - \dfrac{c_i}{2}(a_i^2+u_i^2) + \dfrac{\partial V_i}{\partial S_i}\left(\rho_i u_i\sqrt{S_{3-i}} - \rho_{3-i}u_{3-i}\sqrt{S_i} + \theta_i(k_1 a_1 + k_2 a_2)\right) \\ + \dfrac{\partial V_i}{\partial S_{3-i}}\left(\rho_{3-i}u_{3-i}\sqrt{S_i} - \rho_i u_i\sqrt{S_{3-i}} + \theta_{3-i}(k_1 a_1 + k_2 a_2)\right). \end{cases} \tag{A1}$$

From this, the first-order conditions for u_i and a_i yield, respectively,

$$u_i^* = \frac{\rho_i}{c_i}\left(\frac{\partial V_i}{\partial S_i} - \frac{\partial V_i}{\partial S_{3-i}}\right)\sqrt{S_{3-i}}, \quad a_i^* = \frac{k_i}{c_i}\left(\theta_i\frac{\partial V_i}{\partial S_i} + \theta_{3-i}\frac{\partial V_i}{\partial S_{3-i}}\right). \tag{A2}$$

The first-order conditions for p_1 and p_2 yield

$$1 - 2b_1 p_1 + d_1 p_2 = 0, \quad 1 + d_2 p_1 - 2b_2 p_2 = 0. \tag{A3}$$

Solving the two simultaneous equations in (A3), we obtain the optimal price of firm i to be

$$p_i^* = \frac{d_i + 2b_{3-i}}{4b_1 b_2 - d_1 d_2}. \tag{A4}$$

We can, therefore, simplify the price terms in (A1) so that

$$p_i^*(1 - b_i p_i^* + d_i p_{3-i}^*) = b_i\left(\frac{d_i + 2b_{3-i}}{4b_1 b_2 - d_1 d_2}\right)^2 \equiv m_i. \tag{A5}$$

Henceforth, we will use $m_1 \equiv b_1\left(\frac{d_1 + 2b_2}{4b_1 b_2 - d_1 d_2}\right)^2$ and $m_2 \equiv b_2\left(\frac{d_2 + 2b_1}{4b_1 b_2 - d_1 d_2}\right)^2$ to denote the equilibrium margins of the two firms.

Substituting (A2) and (A5) into (A1) yields

$$\begin{aligned} r_i V_i = m_i S_i &+ \frac{k_i^2}{2c_i}\left(\theta_i\frac{\partial V_i}{\partial S_i} + \theta_{3-i}\frac{\partial V_i}{\partial S_{3-i}}\right)^2 + \frac{\rho_i^2}{2c_i}\left(\frac{\partial V_i}{\partial S_i} - \frac{\partial V_i}{\partial S_{3-i}}\right)^2 S_{3-i} \\ &- \frac{\rho_{3-i}^2}{c_{3-i}}\left(\frac{\partial V_1}{\partial S_1} - \frac{\partial V_1}{\partial S_2}\right)\left(\frac{\partial V_2}{\partial S_2} - \frac{\partial V_2}{\partial S_1}\right)S_i + \frac{k_{3-i}^2}{c_{3-i}}\left(\theta_1\frac{\partial V_1}{\partial S_1} + \theta_2\frac{\partial V_1}{\partial S_2}\right)\left(\theta_1\frac{\partial V_2}{\partial S_1} + \theta_2\frac{\partial V_2}{\partial S_2}\right). \end{aligned} \tag{A6}$$

The linear value function $V_i = \alpha_i + \beta_i S_i + \gamma_i S_{3-i}$ satisfies (A6). The optimal brand and generic advertising decisions in (A2) may now be rewritten as

$$u_i^* = \frac{\rho_i}{c_i}(\beta_i - \gamma_i)\sqrt{S_{3-i}}, \quad a_i^* = \frac{k_i}{c_i}(\theta_i\beta_i + \theta_{3-i}\gamma_i). \tag{A7}$$

Substituting $V_i = \alpha_i + \beta_i S_i + \gamma_i S_{3-i}$ into (A6) and simplifying, we have

$$r_i\alpha_i + r_i\beta_i S_i + r_i\gamma_i S_{3-i} = m_i S_i + \frac{k_i^2}{2c_i}(\theta_i\beta_i + \theta_{3-i}\gamma_i)^2 + \frac{\rho_i^2}{2c_i}(\beta_i - \gamma_i)^2 S_{3-i}$$
$$- \frac{\rho_{3-i}^2}{c_{3-i}}(\beta_1 - \gamma_1)(\beta_2 - \gamma_2)S_i + \frac{k_{3-i}^2}{c_{3-i}}(\theta_1\beta_1 + \theta_2\gamma_1)(\theta_2\beta_2 + \theta_1\gamma_2). \quad \text{(A8)}$$

Equating the coefficients of S_i, S_{3-i}, and the constant in equation (A8) results in the following simultaneous equations to solve for α_1, α_2, β_1, β_2, γ_1, and γ_2:

$$r_i\alpha_i = \frac{k_i^2}{2c_i}(\theta_i\beta_i + \theta_{3-i}\gamma_i)^2 + \frac{k_{3-i}^2}{c_{3-i}}(\theta_1\beta_1 + \theta_2\gamma_1)(\theta_2\beta_2 + \theta_1\gamma_2), \quad i = 1, 2, \quad \text{(A9)}$$

$$r_i\beta_i = m_i - \frac{\rho_{3-i}^2}{c_{3-i}}(\beta_1 - \gamma_1)(\beta_2 - \gamma_2), \quad i = 1, 2, \quad \text{(A10)}$$

$$r_i\gamma_i = \frac{\rho_i^2}{2c_i}(\beta_i - \gamma_i)^2, \quad i = 1, 2. \quad \text{(A11)}$$

Let $y = \beta_1 - \gamma_1$ and $z = \beta_2 - \gamma_2$. Solving for y using *Mathematica v4.0* yields

$$y = -\frac{\eta_2}{4\eta_1} + \frac{1}{2}\left(\sqrt{\omega_1 + \omega_5 + \omega_6} + \sqrt{2\omega_1 - \omega_5 - \omega_6 + \frac{\omega_7}{4\sqrt{\omega_1 + \omega_5 + \omega_6}}}\right), \quad \text{(A12)}$$

where $\eta_1 = 3c_2\rho_1^4$, $\eta_2 = 4c_1c_2\rho_1^2(r_1 + r_2)$, $\eta_3 = 4c_1\left(c_1c_2(2r_1r_2 - r_1^2) + 2c_1m_2\rho_2^2 - c_2m_1\rho_1^2\right)$,

$\eta_4 = 8c_1^2c_2m_1(r_1 - r_2)$, $\eta_5 = 4c_1^2c_2m_1^2$, $\omega_1 = \dfrac{\eta_2^2}{4\eta_1^2} - \dfrac{2\eta_3}{3\eta_1}$, $\omega_2 = \eta_3^2 - 3\eta_2\eta_4 - 12\eta_1\eta_5$,

$\omega_3 = 2\eta_3^3 - 9\eta_2\eta_3\eta_4 + 27\eta_1\eta_4^2 - 27\eta_5\eta_2^2 + 72\eta_1\eta_3\eta_5$, $\omega_4 = (\omega_3 + \sqrt{\omega_3^2 - 4\omega_2^3})^{1/3}$, $\omega_5 = \dfrac{2^{1/3}\omega_2}{3\eta_1\omega_4}$,

$\omega_6 = \dfrac{\omega_4}{32^{1/3}\eta_1}$, and $\omega_7 = \dfrac{4\eta_2\eta_3}{\eta_1^2} - \dfrac{\eta_2^3}{\eta_1^3} - \dfrac{8\eta_4}{\eta_1}$.

Knowing y, we can compute the following:

$$\gamma_1 = \frac{\rho_1^2}{2c_1r_1}y^2,\ \beta_1 = y + \frac{\rho_1^2}{2c_1r_1}y^2,\ z = \frac{c_2}{\rho_2^2 y}(m_1 - \frac{\rho_1^2}{2c_1}y^2 - r_1y),\ \gamma_2 = \frac{\rho_2^2}{2c_2r_2}z^2,\ \beta_2 = \frac{\rho_2^2}{2c_2r_2}z^2 + z. \quad \text{(A13)}$$

One can see from (A13) that $\beta_i > 0$, $\gamma_i > 0$, and $\beta_i > \gamma_i$, resulting in positive values for the controls.

Proof of Corollary 1

Solving the following simultaneous equations for symmetric firms:

$$r\alpha = \frac{3k^2}{8c}(\beta+\gamma)^2, \tag{B1}$$

$$r\beta = m - \frac{\rho^2}{c}(\beta-\gamma)^2, \tag{B2}$$

$$r\gamma = \frac{\rho^2}{2c}(\beta-\gamma)^2, \tag{B3}$$

yields the following solution for γ:

$$\gamma = \frac{3m\rho^2 + cr^2 \pm \sqrt{cr^2(cr^2+6m\rho^2)}}{9r\rho^2}. \tag{B4}$$

To find out which of the two roots to choose, we use the test that $\gamma = 0$ when $m = 0$. This is because the value function should be identically equal to zero when the gross margin is zero since the firm makes zero profit in this case. Checking with (B4), it is easy to see that

$$\gamma = \frac{3m\rho^2 + cr^2 - \sqrt{cr^2(cr^2+6m\rho^2)}}{9r\rho^2} \tag{B5}$$

is the only root that satisfies this condition.

Knowing γ, we can compute β and α using $\beta = \frac{m}{r} - 2\gamma$ and (B1), respectively, to be

$$\beta = \frac{3m\rho^2 - 2cr^2 + 2\sqrt{cr^2(cr^2+6m\rho^2)}}{9r\rho^2}, \tag{B6}$$

$$\alpha = \frac{k^2\left(c^2r^4 - 3cmr^2\rho^2 + 18m^2\rho^4 + (6m\rho^2 - cr^2)\sqrt{cr^2(cr^2+6m\rho^2)}\right)}{108cr^3\rho^4}. \tag{B7}$$

An examination of equations (B6-7) reveals that $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\beta > \gamma$, so the controls and the value functions are positive.

<u>Proof of Proposition 2</u>

To derive the optimal sales paths, we substitute the results from Proposition 1 into the two state equations to obtain the following system of differential equations:

$$\begin{aligned}
\dot{S}_1 &= \frac{\rho_1^2}{c_1}(\beta_1-\gamma_1)S_2 - \frac{\rho_2^2}{c_2}(\beta_2-\gamma_2)S_1 + \theta_1\left(\frac{k_1^2}{c_1}(\theta_1\beta_1+\theta_2\gamma_1) + \frac{k_2^2}{c_2}(\theta_1\gamma_2+\theta_2\beta_2)\right), \quad S_1(0) = S_{10}, \\
\dot{S}_2 &= \frac{\rho_2^2}{c_2}(\beta_2-\gamma_2)S_1 - \frac{\rho_1^2}{c_1}(\beta_1-\gamma_1)S_2 + \theta_2\left(\frac{k_1^2}{c_1}(\theta_1\beta_1+\theta_2\gamma_1) + \frac{k_2^2}{c_2}(\theta_1\gamma_2+\theta_2\beta_2)\right), \quad S_2(0) = S_{20}.
\end{aligned} \tag{C1}$$

For expositional ease, denote

$$\psi_1 = \frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1),\ \psi_2 = \frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2),\ \psi_3 = \frac{k_1^2}{c_1}(\theta_1\beta_1 + \theta_2\gamma_1) + \frac{k_2^2}{c_2}(\theta_1\gamma_2 + \theta_2\beta_2). \quad \text{(C2)}$$

The differential equations can now be rewritten as

$$\begin{aligned} \dot{S}_1 &= \psi_1 S_2 - \psi_2 S_1 + \theta_1\psi_3, \quad S_1(0) = S_{10}, \\ \dot{S}_2 &= \psi_2 S_1 - \psi_1 S_2 + \theta_2\psi_3, \quad S_2(0) = S_{20}. \end{aligned} \quad \text{(C3)}$$

Note from (C3) that there is no long-run equilibrium in sales, i.e., $\dot{S}_1$ and $\dot{S}_2$ need not go to zero. The long-run equilibrium market shares resulting from the equations in (C3) are given by

$$\left(\bar{x}_1 = \lim_{t\to\infty} \frac{S_1(t)}{S_1(t)+S_2(t)}, \quad \bar{x}_2 = \lim_{t\to\infty} \frac{S_2(t)}{S_1(t)+S_2(t)} \right). \quad \text{(C4)}$$

Simplifying, we have

$$\left(\bar{x}_1 = \frac{\frac{\rho_1^2}{c_1}(\beta_1-\gamma_1)}{\frac{\rho_1^2}{c_1}(\beta_1-\gamma_1) + \frac{\rho_2^2}{c_2}(\beta_2-\gamma_2)}, \quad \bar{x}_2 = \frac{\frac{\rho_2^2}{c_2}(\beta_2-\gamma_2)}{\frac{\rho_1^2}{c_1}(\beta_1-\gamma_1) + \frac{\rho_2^2}{c_2}(\beta_2-\gamma_2)} \right). \quad \text{(C5)}$$

<u>Proof of Proposition 3</u>

If the firm owns both brands, its decision problem is

$$\max_{u_1(t),\, a_1(t),\, p_1(t),\, u_2(t),\, a_2(t),\, p_2(t)} V = \int_0^\infty \begin{array}{l} e^{-rt}\big(p_1(t)S_1(t)(1-b_1p_1(t)+d_1p_2(t)) + p_2(t)S_2(t)(1+d_2p_1(t)-b_2p_2(t)) \\ \qquad -\frac{c_1}{2}(a_1(t)^2+u_1(t)^2) - \frac{c_2}{2}(a_2(t)^2+u_2(t)^2)\big)dt, \end{array} \quad \text{(D1)}$$

$$\text{s.t.} \quad \begin{aligned} \dot{S}_1(t) &= \rho_1u_1(t)\sqrt{S_2(t)} - \rho_2u_2(t)\sqrt{S_1(t)} + \theta_1\big(k_1a_1(t)+k_2a_2(t)\big), \quad S_1(0) = S_{10}, \\ \dot{S}_2(t) &= \rho_2u_2(t)\sqrt{S_1(t)} - \rho_1u_1(t)\sqrt{S_2(t)} + \theta_2\big(k_1a_1(t)+k_2a_2(t)\big), \quad S_2(0) = S_{20}, \end{aligned} \quad \text{(D2)}$$

where the notation is as described earlier.

The HJB equation is

$$rV = \max_{u_1,\, a_1,\, p_1,\, u_2,\, a_2,\, p_2} \begin{cases} p_1S_1(1-b_1p_1+d_1p_2) + p_2S_2(1+d_2p_1-b_2p_2) - \frac{c_1}{2}(u_1^2+a_1^2) - \frac{c_2}{2}(u_2^2+a_2^2) \\ +\frac{\partial V}{\partial S_1}\big(\rho_1u_1\sqrt{S_2} - \rho_2u_2\sqrt{S_1} + \theta_1(k_1a_1+k_2a_2)\big) \\ +\frac{\partial V}{\partial S_2}\big(\rho_2u_2\sqrt{S_1} - \rho_1u_1\sqrt{S_2} + \theta_2(k_1a_1+k_2a_2)\big). \end{cases} \quad \text{(D3)}$$

The first-order conditions for the optimal advertising decisions yield

$$u_1^* = \frac{\rho_1}{c_1}(\frac{\partial V}{\partial S_1} - \frac{\partial V}{\partial S_2})\sqrt{S_2}, \quad a_1^* = \frac{k_1}{c_1}(\theta_1 \frac{\partial V}{\partial S_1} + \theta_2 \frac{\partial V}{\partial S_2}), \tag{D4}$$

$$u_2^* = \frac{\rho_2}{c_2}(\frac{\partial V}{\partial S_2} - \frac{\partial V}{\partial S_1})\sqrt{S_1}, \quad a_2^* = \frac{k_2}{c_2}(\theta_1 \frac{\partial V}{\partial S_1} + \theta_2 \frac{\partial V}{\partial S_2}). \tag{D5}$$

As before, substituting the solutions from (D4-5) into (D3) suggests that a linear value function $V = \alpha_m + \beta_m S_1 + \gamma_m S_2$ will solve the resulting partial differential equation. The optimal advertising decisions can now be rewritten as

$$u_1^* = \max\{0, \ \frac{\rho_1}{c_1}(\beta_m - \gamma_m)\sqrt{S_2}\}, \quad a_1^* = \frac{k_1}{c_1}(\theta_1\beta_m + \theta_2\gamma_m), \tag{D6}$$

$$u_2^* = \max\{0, \ \frac{\rho_2}{c_2}(\gamma_m - \beta_m)\sqrt{S_1}\}, \quad a_2^* = \frac{k_2}{c_2}(\theta_1\beta_m + \theta_2\gamma_m). \tag{D7}$$

Note from (D6-7) that either u_1^* or u_2^* is always positive. If $\beta_m > \gamma_m$, $u_1^* > 0$ and $u_2^* = 0$ since brand 1 is more profitable. If $\beta_m < \gamma_m$, the opposite is true. Therefore, in a monopoly, total brand advertising need not necessarily be zero.

The monopolist can choose the optimal advertising decisions to ensure the value function in the monopoly case is never less than that in the competitive one. In other words, $V \geq V_1 + V_2$, where V_1 and V_2 are the profits in the competitive case. We, therefore, have

$$\alpha_m + \beta_m S_1 + \gamma_m S_2 \geq \alpha_1 + \beta_1 S_1 + \gamma_1 S_2 + \alpha_2 + \gamma_2 S_1 + \beta_2 S_2, \tag{D8}$$

$$\alpha_m - (\alpha_1 + \alpha_2) + (\beta_m - (\beta_1 + \gamma_2))S_1 + (\gamma_m - (\beta_2 + \gamma_1))S_2 \geq 0. \tag{D9}$$

Since equation (D9) holds $\forall \ S_1 \geq 0, \ S_2 \geq 0$, it must be the case that

$$\alpha_m \geq (\alpha_1 + \alpha_2), \ \beta_m \geq \beta_1 + \gamma_2, \ \gamma_m \geq \beta_2 + \gamma_1, \tag{D10}$$

where each of the above coefficients is non-negative.

From equation (A7), the total generic advertising in the competitive case is

$$\frac{k_1}{c_1}(\theta_1\beta_1 + \theta_2\gamma_1) + \frac{k_2}{c_2}(\theta_1\gamma_2 + \theta_2\beta_2), \tag{D11}$$

while that in the monopoly case is, from equations (D6-7),

$$\frac{k_1}{c_1}(\theta_1\beta_m + \theta_2\gamma_m) + \frac{k_2}{c_2}(\theta_1\beta_m + \theta_2\gamma_m). \tag{D12}$$

Subtracting equation (D11) from (D12), the difference between the total generic advertising in the monopoly case and that in the competitive case is

$$\frac{k_1}{c_1}(\theta_1(\beta_m - \beta_1) + \theta_2(\gamma_m - \gamma_1)) + \frac{k_2}{c_2}(\theta_1(\beta_m - \gamma_2) + \theta_2(\gamma_m - \beta_2)), \qquad \text{(D13)}$$

which, from equation (D13), is greater than zero. Therefore, the monopolist's total generic advertising is greater than that under competition.

Table 1: *Comparison of Dynamic Competitive Models of Brand Advertising*

Study	Market Expansion	Generic Advertising	Solution Procedure		
			Closed-loop	Explicit Solutions for Closed-loop	Remarks
Chintagunta and Vilcassim 1992	×	×	√	√	Solutions are obtained only for the case of no discounting.
Chintagunta 1993	×	×	√	√	Explicit solutions are derived for a two-period model with symmetric competitors.
Chintagunta and Jain 1995	×	×	√	N/A	The purpose of the study was econometric estimation to test the Sorger (1989) specification.
Deal 1979	×	×	×	×	Numerical techniques are used to obtain the optimal open-loop advertising decisions.
Deal, Sethi, and Thompson 1979	×	×	×	×	Numerical techniques are used to obtain the optimal open-loop advertising decisions.
Erickson 1985	√	×	×	×	Numerical techniques are used to obtain the optimal open-loop advertising decisions.
Erickson 1992	×	×	√	√	Explicit solutions are obtained only for the case of no discounting.
Erickson 1995	√	×	√	×	Numerical techniques are used to obtain the optimal closed-loop advertising decisions.
Espinosa and Mariel 2001	√	×	√	√	Explicit solutions are presented only for symmetric firms. Even so, these solutions are obtained for pure informative and pure predatory advertising, and when the sales level of one firm is independent of the rival's advertising.
Fruchter and Kalish 1997	×	×	√	√	Explicit solutions are obtained assuming the two firms have the same discount rate.
Fruchter 1999	√	×	√	√	Explicit solutions are obtained only for the case of no discounting.
Horsky 1977	×	×	×	N/A	The purpose of the study was econometric estimation of the sales response function.
Horsky and Mate 1988	√	×	√	×	Numerical techniques are used to obtain the optimal closed-loop advertising decisions.
Jorgensen 1982b	×	×	×	√	Explicit solutions are obtained assuming the two firms have the same discount rate.
Piga 1998	√	×	√	√	Explicit solutions are obtained assuming the two firms are symmetric.
Roberts and Samuelson 1988	√	×	√	N/A	The purpose of the study was econometric estimation of the sales response function.
Sorger 1989	×	×	√	√	Explicit solutions are obtained assuming a mature market.
Current Study	√	√	√	√	**Explicit solutions are obtained for the general model.**

Table 2: *Comparative Statics Results for the Symmetric Case*

Variables	c	m	k	ρ	r
α	?	↑	↑	↓	↓
β	↑	↑	=	↓	↓
γ	↓	↑	=	↑	↓
u_i^*	↓	↑	=	↑	↓
a_i^*	↓	↑	↑	↓	↓
V_i	?	↑	↑	?	↓
$\overline{x}_i$	↓	↑	=	↑	↓

Legend: ↑ increase; ↓ decrease; = unchanged; ? ambiguous.

Table 3: *Comparative Statics Results for the Asymmetric Case*

Variables	c_i	c_{3-i}	m_i	m_{3-i}	k_i	k_{3-i}	ρ_i	ρ_{3-i}	r_i	r_{3-i}	θ_i
α_i	?	?	?	?	↑	↑	?	?	?	?	?
β_i	?	↑	↑	↓	=	=	?	↓	↓	↑	=
γ_i	↓	↑	↑	↓	=	=	↑	↓	↓	↑	=
u_i^*	↓	↑	↑	↓	=	=	↑	↓	↓	↑	=
a_i^*	↓	↑	↑	↓	↑	=	?	↓	↓	↑	↑
V_i	?	?	?	?	↑	↑	?	?	?	?	?
$\overline{x}_i$	↓	↑	↑	↓	=	=	↑	↓	↓	↑	=

Legend: ↑ increase; ↓ decrease; = unchanged; ? ambiguous.

Figure 1: *Optimal Generic and Brand Advertising for the Same Advertising Budget*

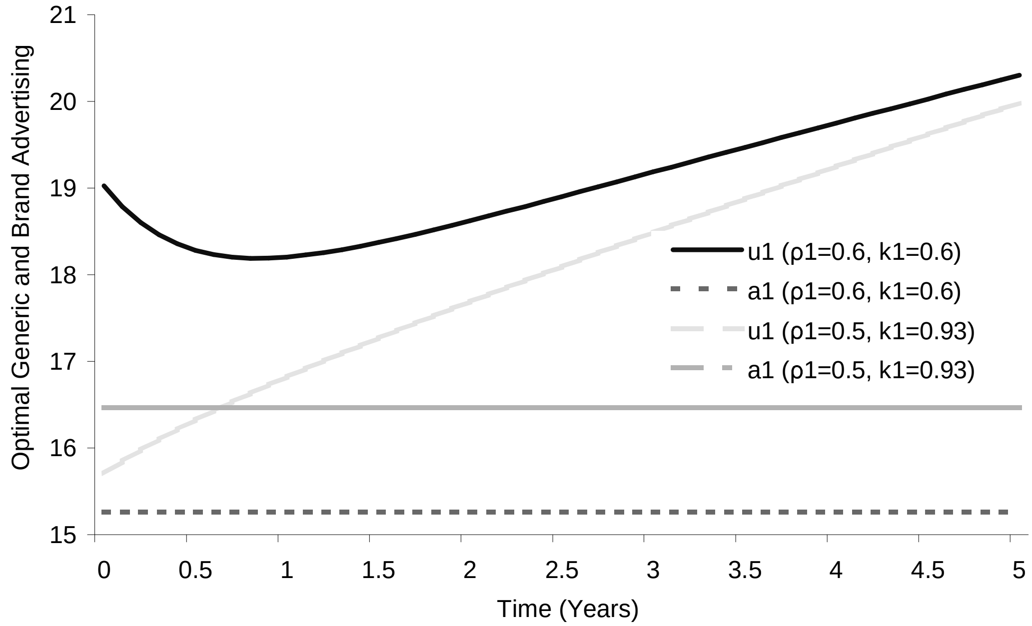
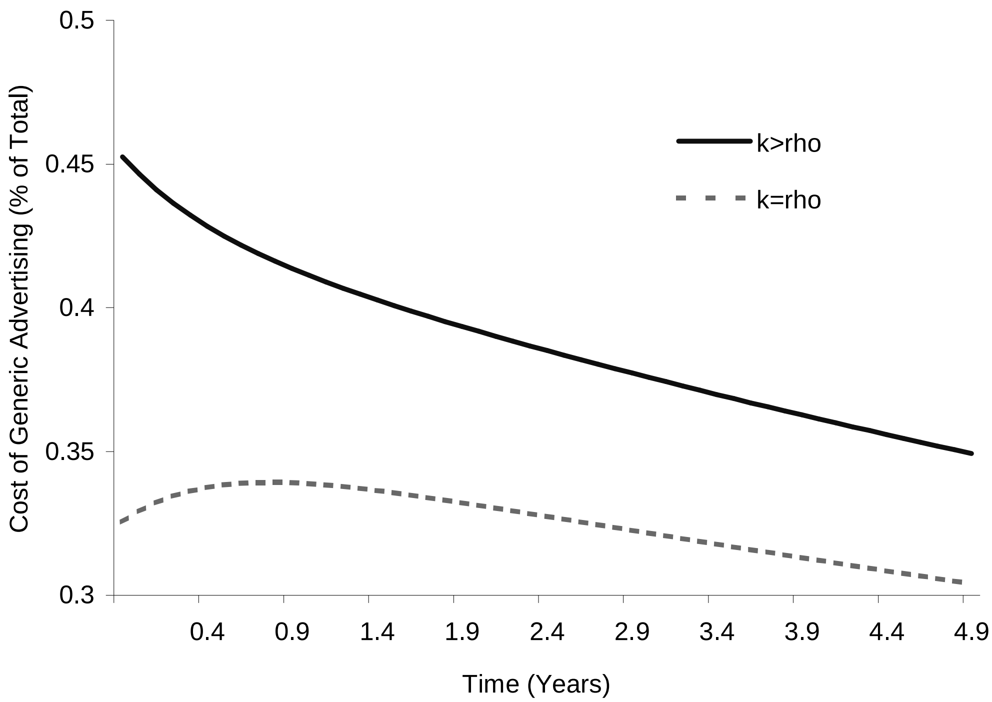


Figure 2: *Proportion of Optimal Generic and Brand Advertising*



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