Logic / Discrete Math - Practice Final

PART 1 - will be worth 100 POINTS on the exam

1. Demonstrate $A \to B, B \to \sim C, B \vee C, \sim B \vdash \sim A$

2. Demonstrate using RCP: $P \to S, (Q \wedge S) \to (R \wedge T) \vdash (P \wedge Q) \to R$

3. Demonstrate using IP:
$\forall x \exists y \left\{ P(x,y) \to Q(x,y) \right\}, \ \forall x \forall y P(x,y) \ \vdash \ \forall x \exists y Q(x,y)$

4. Demonstrate: $\vdash A \cap B = \emptyset \to A \subset \overline{B}$. Use no Theorems as reasons. You *may* use Axioms, and Definitions as reasons.

5(Logic). Give an informal proof of : if $A \cap B = \emptyset$, then $A \subset \overline{B}$.

5(Discrete). Suppose A is the alphabet $\{x, y\}$ and S is the language $\{\lambda, x, y, yx, xyx\}$.

a) True or False:

______ $\lambda \in A^2$

______ $S \subset A^*$

______ $S \subset (A^2)^*$

b) How many elements of length 3 are there in S^2? **Ans:**______

List them here:

c) Prove by example that A^* is not a subset of $(A^2)^*$

PART 2 - will be worth 100 POINTS on the exam

NOTE: All proofs in Problems 6-12 are informal.

6. Use PMI and Th. 4.22 to prove:
For all $n \geq 2,\ A_1 \cap A_2 \cap \cdot\cdot\cdot \cap A_n = A_n \cap A_{n-1} \cap \cdot\cdot\cdot \cap A_1$

7. Prove by Mathematical Induction (PMI): $\forall n \geq 2$, $2^{2n} < (n+2)!$

8. A sequence a_n is defined: $a_0 = 3,\ a_1 = 1,\ a_{n+2} = 2a_{n+1} + 3a_n$ for $n \geq 0$. Use the Fibonacci Variation to prove that $a_n = 3^n + 2(-1)^n$ for all $n \geq 0$.

9. Prove F = $\left\{ \Big((x,y), z \Big) : x,y,z \in \mathbb{R} \text{ and } (x+y)z - 4z - 1 = 0 \right\}$ is a function.

10. Let G = $\left\{ (x,y) : x,y \in \mathbb{R} \text{ and } y^2 - 5x = 5 \right\}$

a) Prove G is <u>not</u> a function.

b) True or false (same G as above):

1) Domain of G = $[0, \infty)$ **Ans:**______

2) Range of G = $\mathbb{R}$ **Ans:**______

11. Suppose F and G are functions such that D(F) = D(G) = $\mathbb{R}$ and

i) $\forall x \in \mathbb{R}\quad F(x) = x - |x+1|$ ii) $G(x) = \begin{cases} -1 \text{ if } x \geq -1 \\ 2x+1 \text{ if } x < -1 \end{cases}$.

Use a "proof by cases" to prove that F = G.

12. Let F be a function such that: $D(F) = \mathbb{R}$ and $\forall x \in \mathbb{R}$, $F(x) = x^2 - 5$.
Prove: $F : \mathbb{R} \underset{\text{onto}}{\rightarrow} [-5, \infty)$.

Hints and Answers for Practice Final

1. Hint: Can be done with DS, HS and MT.
2. Assume P $\wedge$ Q, then use S twice
3. Begin with the assumption $\sim \forall x \exists y Q(x, y)$. Watch out for the oath!
4. Assume A $\cap$ B = $\emptyset$, then assume $a \in$ A. A4 can then be used to show $a \notin$ B.
5L. Begin: Assume A $\cap$ B = $\emptyset$ and let x be an arb. elt. of A.
5D. a) F, T, F
b) There are 4 elements of length 3 in S^2. They are: xyx, y^2x, yx^2, yxy
c) Note $x \in A^*$, but not to $(A^2)^*$, since all elements of $(A^2)^*$ have even length.

6. Let P(n) be the statement $A_1 \cap A_2 \cap \cdots \cap A_n = A_n \cap A_{n-1} \cap \cdots \cap A_1$.
Base step - P(2) : $A_1 \cap A_2 = A_2 \cap A_1$ (?). True by T4.22.
Ind. hyp. - Assume P(k) : $A_1 \cap A_2 \cap \cdots \cap A_k = A_k \cap A_{k-1} \cap \cdots \cap A_1$.
Ind. step - Prove P$(k+1)$: LHS = $A_1 \cap A_2 \cap \cdots \cap A_{k+1} = A_{k+1} \cap A_k \cap \cdots \cap A_1$ = RHS
Proof: LHS = $A_1 \cap A_2 \cap \cdots \cap A_{k+1} = (A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1}$
$= A_{k+1} \cap (A_1 \cap A_2 \cap \cdots \cap A_k)$ (by T4.22)
$= A_{k+1} \cap (A_k \cap A_{k-1} \cap \cdots \cap A_1)$ (by ind. hyp.)
$= A_{k+1} \cap A_k \cap \cdots \cap A_1$
$=$ RHS
Thus P$(k+1)$ is true. By PMI, P(n) is true for all $n \geq 2$.

7. Let P(n) be the statement $2^{2n} < (n+2)!$
Base step - P(2) : $2^4 < 4!$ (?) True by calculation, since $16 < 24$.
Ind. hyp. - Assume P(k) : $2^{2k} < (k+2)!$
Ind. step - Prove P$(k+1)$: LHS = $2^{2k+2} < (k+3)!$ = RHS
Proof: LHS = $2^{2k+2} = 4 \cdot 2^{2k} \underset{\text{(by ind hyp)}}{<} 4(k+2)! \underset{\text{(since } k \geq 2)}{<} (k+3)(k+2)! =$ $(k+3)! =$ RHS. Thus P$(k+1)$ is true. By PMI, P(n) is true for all $n \geq 2$.

8. Let P(n) be the statement $a_n = 3^n + 2(-1)^n$.
Base step - P(0) : $a_0 = 3^0 + 2(-1)^0$. True by calculation, since $a_0 = 3$ and $3^0 + 2(-1)^0 = 3$.
P(1) : $a_1 = 3^1 + 2(-1)^1$. True by calculation, since $a_1 = 1$ and $3^1 + 2(-1)^1 = 1$.
Ind. hyp. - Assume P(k) and P$(k+1)$ are true; i.e.,
$a_k = 3^k + 2(-1)^k$ and $a_{k+1} = 3^{k+1} + 2(-1)^{k+1}$.
Ind. step - Prove P$(k+2)$: LHS = $a_{k+2} = 3^{k+2} + 2(-1)^{k+2}$ = RHS.
Proof: LHS = $a_{k+2} = 2a_{k+1} + 3a_k$

$$= 2\left(3^{k+1} + 2(-1)^{k+1}\right) + 3\left(3^k + 2(-1)^k\right) \text{ (by ind. hyp)}$$
$$= (2 \cdot 3^{k+1} + 3^{k+1}) + 4(-1)^{k+1} + 6(-1)^k \text{ (by alg. regrouping)}$$
$$= 3 \cdot 3^{k+1} - 4(-1)^{k+2} + 6(-1)^{k+2} \text{ (since } (-1)^k = (-1)^{k+2})$$
$$= 3^{k+2} + 2(-1)^{k+2}$$
$$= \text{RHS}$$

Thus P$(k+2)$ is true. By PMI, P(n) is true for all $n \geq 0$.

9. Key step: note that $x + y - 4 \neq 0$. Otherwise, the equation $(x+y)z - 4z - 1 = 0$ becomes $-1 = 0$, a contradiction. Thus we may solve for z and w: $z = \frac{1}{x+y-4} = z$.

10. a) Note that $(0, \sqrt{5}) \in$ G and $(0, -\sqrt{5}) \in$ G and $\sqrt{5} \neq -\sqrt{5}$.
b) 1) F, 2) T

11. Note that D(F) = D(G) is given.
Let x be an arb. elt. of $\mathbb{R}$.
Case 1. Let $x \geq -1$. Then G$(x) = -1$ (given) and F$(x) = x - (x+1) = -1$. Thus F$(x) =$ G(x).
Case 2. Let $x < -1$. Then G$(x) = 2x + 1$ (given) and F$(x) = x - [-(x+1)] = 2x + 1$. Thus F$(x) =$ G(x).
Thus F$(x) =$ G(x) for all x in $\mathbb{R}$, so F = G.

12. We must show $[-5,\infty) =$ R(F).
$\subset$: Let y be an arb. elt. of $[-5,\infty)$ and let $x = \sqrt{y+5}$. Then $x \in \mathbb{R}$, since $y \geq -5$.
Also, F$(x) =$ F$\left(\sqrt{y+5}\right) = \left(\sqrt{y+5}\right)^2 - 5 = y$. Thus $y \in$ R(F). Since y was arb., $[-5,\infty) \subset$ R(F).
$\supset$: Let y be an arb. elt. of R(F).Then there is an x s.t. $(x, y) \in$ F. Then $x, y \in \mathbb{R}$ and $y = x^2 - 5$. Then $y \geq -5$, since $x^2 \geq 0$. That is, $y \in [-5,\infty)$. Since y was arb., R(F) $\subset [-5,\infty)$.
By double subset, $[-5,\infty) =$ R(F) and thus F is onto.