Example 3 Do an informal proof of: If $A \cap B = B$, then $B \subset A$.
Proof: Assume $A \cap B = B$ and let x be an arb elt of B. Since $A \cap B = B$, x is also in $A \cap B$. That is, $x \in A$ and $x \in B$. In particular, $x \in A$. Thus every elt of B is also in A, so $B \subset A$.□

Example 4: T4.31 $\vdash (A \subset B \wedge C \subset D) \rightarrow A \cap C \subset B \cap D$
(Informal statement: If $A \subset B$ and $C \subset D$, then $A \cap C \subset B \cap D$.)
Give a demonstration and a proof *using no Theorems as reasons*.

Formal demonstration:

1. $A \subset B \wedge C \subset D$
2. $A \subset B$
3. $C \subset D$
4. $a \in A \cap C$
5. $a \in A \wedge a \in C$
6. $a \in A$
7. $a \in C$
8. $\forall x(x \in A \rightarrow x \in B)$
9. $a \in A \rightarrow a \in B$
10. $a \in B$
11. $\forall x(x \in C \rightarrow x \in D)$
12. $a \in C \rightarrow a \in D$
13. $a \in D$
14. $a \in B \wedge a \in D$
15. $a \in B \cap D$
16. $a \in A \cap C \rightarrow a \in B \cap D$
17. $\forall x(x \in A \cap C \rightarrow x \in B \cap D)$
18. $A \cap C \subset B \cap D$
19. $(A \subset B \wedge C \subset D) \rightarrow A \cap C \subset B \cap D$

Informal proof:

Assume $A \subset B$ and $C \subset D$ and let x be an arb elt of $A \cap C$. This means $x \in A$ and $x \in C$. But, since $x \in A$ and $A \subset B$, then $x \in B$. Similarly, $x \in D$. Thus, $x \in B \cap D$. Therefore, every elt. of $A \cap C$ is also in $B \cap D$, so $A \cap C \subset B \cap D$.□

Homework (with some hints):

EXERCISE 38 Do an informal proof of: $A \subset A \cup B$. (T4.7)
Hint - begin: Let x be an arb elt of A.

EXERCISE 39 Do an informal proof of: $A \cap U = A$. (T4.12)
Hint - double subset.

EXERCISE 40 Do an informal proof of: If $A \cup B \subset B$, then $A \subset B$.

EXERCISE 41 Do a demonstration <u>and</u> an informal proof of: If $A \cup B = A \cap B$, then $A \subset B$.
Hint - begin: Assume $A \cup B = A \cap B$ and let x be an arb elt of A. Reread Example 3.

EXERCISE 42 Do an informal proof of: If $A \subset B$, then $A \cap C \subset B \cap C$.

Example 5 Do a demonstration <u>and</u> an informal proof of: If $A \cap B = U$, then $A = U$.
Proof: Assume $A \cap B = U$ and let x be an arb elt of U. Then the assumption says $x \in A \cap B$. That is, $x \in A$ and $x \in B$. In particular, $x \in A$. Thus every elt of U is in A, so $A = U$. □

Example 6. Do an informal proof of: If $\overline{A} \cup B = U$, then $A \subset B$.
Proof: Assume $\overline{A} \cup B = U$ and let x be an arb elt of A. Since $\overline{A} \cup B = U$ and $x \in U$, we have $x \in \overline{A} \cup B$. That is, $x \notin A$ or $x \in B$. Since $x \in A$, this forces $x \in B$. Thus every elt of A is also in B, so $A \subset B$. □
(Note: do you see a Disjunctive Syllogism in this argument?)

Example 7. Give a demonstration and a proof **using no Theorems as reasons** for:
T4.13 $\vdash A \cap \overline{A} = \emptyset$

Formal demonstration:

1. $\quad| \exists x(x \in A \cap \overline{A})$
2. $\quad| a \in A \cap \overline{A}$
3. $\quad| a \in A \wedge a \in \overline{A}$
4. $\quad| a \in A \wedge a \notin A$
5. $\exists x(x \in A \cap \overline{A}) \rightarrow (a \in A \wedge a \notin A)$
6. $\sim \exists x(x \in A \cap \overline{A})$
7. $\forall x(x \notin A \cap \overline{A})$
8. $A \cap \overline{A} = \emptyset$

Informal proof:

We wish to show no elt of U can belong to $A \cap \overline{A}$. Suppose $A \cap \overline{A}$ contains at least one element, x. Then $x \in A$ and $x \in \overline{A}$; i.e. $x \in A$ and $x \notin A$, a contradiction. Thus no such x can exist, i.e. $A \cap \overline{A} = \emptyset$. □

EXERCISE 43 Do a demonstration <u>and</u> an informal proof of: If $A \cap B = \emptyset$, then $A \subset \overline{B}$.
Hint - use an indirect argument near the middle of your proof (let $x \in B, etc$).

EXERCISE 44 Do an informal proof of: If $A \subset \overline{A} \cup B$, then $A \subset B$.
Hint - uses some reasoning similar to Example 6 above.

EXERCISE 45 Do an informal proof of: If $A \cup \overline{B} \subset B \cup \overline{A}$, then $A \subset B$.

EXERCISE 46 Do an informal proof of: If $A \cup B = \emptyset$, then $A = \emptyset$.
Hint - begin: Assume $A \cup B = \emptyset$ and suppose A contains at least one element, x. Then reach a contradiction.

EXERCISE 47 Do an informal proof of: If $A \cup B \subset \overline{C}$, then $A \cap C = \emptyset$.

Example 8. (Example of “proof by cases”) We will prove: $A \subset B \vdash \overline{A} \cup B = U$
Note the “premise $\vdash$ conclusion” form.

Formal demonstration:

1. $A \subset B$
2. $\quad | \; a \in A$
3. $\quad | \; \forall x(x \in A \to x \in B)$
4. $\quad | \; a \in A \to a \in B$
5. $\quad | \; a \in B$
6. $\quad | \; a \in \overline{A} \vee a \in B$
7. $\quad | \; a \in \overline{A} \cup B$
8. $a \in A \to a \in \overline{A} \cup B$
9. $\quad | \; a \notin A$
10. $\quad | \; a \in \overline{A}$
11. $\quad | \; a \in \overline{A} \vee a \in B$
12. $\quad | \; a \in \overline{A} \cup B$
13. $a \notin A \to a \in \overline{A} \cup B$
14. $a \in A \vee a \notin A$
15. $a \in \overline{A} \cup B \vee a \in \overline{A} \cup B$
16. $a \in \overline{A} \cup B$

17 $\forall x\left(x \in \overline{A} \cup B\right)$

18. $\overline{A} \cup B = U$

Informal proof:

Assume $A \subset B$. We wish to show every element of U belongs to $\overline{A} \cup B$. Let x be an arb. elt. of U. Then $x \in A$ or $x \notin A$.
Case 1: let $x \in A$. Since $A \subset B$, $x \in B$. Thus $x \in \overline{A}$ or $x \in B$; i.e., $x \in \overline{A} \cup B$.
Case 2: let $x \notin A$. Then $x \in \overline{A}$; i.e., $x \in \overline{A} \cup B$
Thus every elt. of U is in $\overline{A} \cup B$, i.e
$\overline{A} \cup B = U$. ☐

Note: the informal proof began with the word 'assume' even though the demonstration did not begin with an assumption.

Miscellaneous Informal Proofs

EXERCISE 48 For each argument form, first do a demonstration (or find one in your notes), *using no Theorems as reasons*. Then do an informal proof, using your demonstration as a guide.
a) $\vdash (A \subset B \wedge A \subset C) \to A \subset B \cap C$
b) T4.14 $\vdash A \cup \overline{A} = U$
c) $A \subset B \vdash A \cap \overline{B} = \emptyset$
d) $A \subset B \vdash A \subset A \cap B$

EXERCISE 49 In these, again *using no Theorems as reasons*, try to do only an informal proof.
a) If $A \cup B \subset C$, then $(A \subset C$ and $B \subset C)$
b) If $A \subset B \cup C$ and $A \cap C = \emptyset$, then $A \subset B$
c) If $A \subset B$ and $B \subset C$, then $A \subset B \cap C$
d) If $A \cup B = A$, then $B \subset A$

Strings

Definitions An *alphabet* A is any non-empty set of symbols. For example, $A = \{a, b\}$ is a typical alphabet we will work with. A *string over* A is an object formed by the juxtaposition (i.e., placing side-by-side) of a finite number of elements from A. For example, $v = a$, $v = ab$, $v = ba$, $v = ababa$, $v = aaabb$ are five different strings over $A = \{a, b\}$. (Two strings are equal if and only if they match character by character.)

Comments 1) There is no "commutative property" for strings. For example, if a and b are elements of the alphabet, ab and ba are different strings. However, there is an "associative property": $a(ba) = (ab)a = aba$ (i.e., the parentheses are not required).
2) A string like aaa can, and usually will, be written as a^3. For example, the string $aaabb$ above can be written a^3b^2. Also, $(ab)^2 = abab$.

Definition There is a *null string*, λ, consisting of no symbols. A string of the form a^0 is considered to be the null string. The null string λ has the property that, for any string v, $\lambda v = v\lambda = v$.

Definition If v and w are strings over A, we can form a string vw as v followed by w (this is sometimes called the *concatenation* of v and w). In particular, the string v^2 means vv (v followed by v).

For example, if $v = a^2$ and $w = ab^2a$, then $vw = (a^2)(ab^2a) = a^3b^2a$ and $v^2 = a^4$.

Question: If v and w are strings over A, is $vw \neq wv$?

Definition The *length* of a string v, denoted $|v|$, is the total number of symbols from A that make up v, including repetitions. The length of the null string λ will be defined to be 0.

For example, with v and w as above, $|vw| = |a^3b^2a| = 6$.

Comments Observe that, in general, $|vw| = |v| + |w|$. Also, since $|\lambda| = 0$, λ is never counted in $|v|$. For example, if a is an element of our alphabet, $|\lambda a| = |\lambda| + |a| = 0 + 1 = 1$.

Definition Any set of strings over the alphabet A is called a $language$ over A.

Example Let $A = \{a, b\}$. What is the set $S = \{v : v \text{ is a string over } A \text{ and } |v| = 3\}$? This is the set of all strings over A with length 3. Let's list them: $S = \{aaa, aab, aba, baa, abb, bab, bba, bbb\} = \{a^3, a^2b, aba, ba^2, ab^2, bab, b^2a, b^3\}$. Note the pattern in this list: first those strings with 3 a's, followed by 2 a's and 1 b, etc.

EXERCISE 50 Simplify the following strings, using exponents whenever possible:
a) $a\lambda a$
b) $a(a\lambda b)(ab\lambda)(\lambda ab)b$
c)$a^2\lambda^3b^2\lambda(ab\lambda)^4$.
EXERCISE 51 Give the length of each of the three strings in Exercise 50.
EXERCISE 52 List the elements of $T = \{v : v$ is a string over A and $|v| \leq 3\}$ for $A = \{a, b\}$.
EXERCISE 53 Explain how we can count the number of strings of length n over $A = \{a, b\}$. What about the number of string of length $\leq n$?

Definition For any alphabet A and $n \geq 0$, we define $A^n = \{v : v$ is a string over A and $|v| = n\}$. Thus the set S from the above Example is A^3. Note that $A^0 = \{\lambda\}$.

Comment Note that, for $n \geq 1$, A^n can also be expressed in the following form using the notion of concatenation mentioned above: $A^n = \{a_1a_2 \cdot\cdot\cdot a_n : a_i \in A$ for $i = 1, 2, \cdot\cdot\cdot, n.\}$.

Definition Using the idea in the above comment, we can define S^n (where $n \geq 1$) for any set of strings S over A: $S^n = \{x_1x_2 \cdot\cdot\cdot x_n : x_i \in S$ for $i = 1, 2, \cdot\cdot\cdot, n.\}$. S^0 is defined to be $\{\lambda\}$.

Question: Let a be a symbol and n a positive integer. What is $\{a\}^n$?

Definition Consider two alphabets, A and B (possibly the same), and suppose S is a set of strings over A and T is a set of strings over B. We define $ST = \{vw : v \in S$ and $w \in T\}$.

Example Let $S = \{\lambda, a^2, b\}$ and $T = \{a, b^2\}$. Then $ST = \{a, b^2, a^3, a^2b^2, ba, b^3\}$.
Note that $ab \in TS$, but $ab \notin ST$. This is a "proof by example" that $ST \neq TS$.

EXERCISE 54 List the elements of A^0, A, A^2, and A^3 for $A = \{a, b\}$.
EXERCISE 55 Let $A = \{a, b\}$ and $S = \{v : v$ is a string over A and $2 \leq |v| \leq 3\}$. Express S as a union of powers of A.
EXERCISE 56 Let $A = \{a\}$ and $S = \{v : v$ is a string over A and $|v| \leq 100\}$. Express S as a union of powers of A and list the elements of S. (Use the ". . ." notation.)
EXERCISE 57 Let $A = \{a, b\}$ and $S = \{\lambda, a, b\}$. Prove by example that $S^3 \neq A^3$.
EXERCISE 58 Consider the language $T = \{a, b, ab, ba\}$ over the alphabet $A = \{a, b\}$.
a) $T^2 =$?
b) Prove by example that $|T^2| < |T|^2$ (Notation: for any set S, $|S| =$ number of elements in S.)
c) Given a non-empty <u>alphabet</u> A, what is the relationship between $|A^2|$ and $|A|^2$ ($=$, $<$, $>$)? Explain.
EXERCISE 59 Let $A = \{a, b\}$ and $B = \{b, c\}$. a) List all the elements of AB^2.
b) Let v be an element of A^3B^3. What can be said about $|v|$?
EXERCISE 60 a) Suppose $A = \{a, b\}$ and $S = \{a, b, ab\}$. What is S^2?
b) Find an element that belongs to $S^3 \cap S^4$.

Challenge problem 1 Let A be an alphabet and S be a non-empty language over A. Prove: $S \subset S^2$ if and only if $\lambda \in S$.

Definition Let A be a non-empty alphabet. The set A^* (often referred to as the "Kleene Star of A") is defined as the set of all strings over A (including the null string λ). For example, if a is a symbol, $\{a\}^* = \{\lambda, a, a^2, a^3, \cdots\} = \{a^n : n \geq 0\}$. It should be clear that A^* has infinitely many elements if A is not empty. For completeness, we define $\emptyset^* = \{\lambda\}$. For a general alphabet A, $A^* = \{\lambda\} \cup A \cup A^2 \cup \cdots$.

Definition Let A be an alphabet. A *language* over A is any set of strings over A (that is, any subset of A^*). For any language S, we may define the Kleene Star of S as $S^* = \{\lambda\} \cup S \cup S^2 \cup \cdots$.

Example If $A = \{a\}$ and $B = \{b\}$, then $A^*B^* = \{a^m b^n : m, n \geq 0\}$. In particular, note that $ab \in A^*B^*$ but $ba \notin A^*B^*$. Also, $AB = \{ab\}$, $A^2B^2 = \{a^2b^2\}$, $(AB)^* = \{(ab)^n : n \geq 0\}$ and $A^nB^n = \{a^n b^n\}$. What is $\bigcup_{n=0}^{\infty} A^n B^n$?

Example Let $A = \{a, b\}$ and $B = \{b, c\}$. Note that the string bc belongs to both A^*B^* and B^*A^*, since $bc = (b)(c)$ (and thus belongs to A^*B^*) and $bc = (bc)(\lambda)$ (and thus belongs to B^*A^*).

EXERCISE 61 Suppose $A = \{a, b\}$ and $B = \{b, c\}$ and consider the strings $a, b, c, ab, ba, bc, cb, ac, ca, abc, ab^2, a^2c, c^2b$.

a) Which of these strings belong to $A^* \cup B^*$?
d) Which ones belong to $(A \cup B)^*$?
c) Which ones belong to A^*B^*?
d) Which ones belong to B^*A^*?

EXERCISE 62 For any alphabet A, what is $(A^*)^2$? How does it compare with $(A^2)^*$?

EXERCISE 63 For any two alphabets A and B, prove that $A^*B^* \cap B^*A^*$ contains at least one element other than λ. (This will be another proof by example.)

EXERCISE 64 a) Let $A = \{a, b\}$ and $B = \{b, c\}$. Prove by example that $A^* \cup B^* \neq (A \cup B)^*$.
b) Use a standard subset proof to show: for any two alphabets A and B, $A^* \cup B^* \subset (A \cup B)^*$.

EXERCISE 65 a) Prove: If S is a language over A, then $S^* \subset A^*$.
b) Prove: if S and T are languages such that $S \subset T$, then $S^* \subset T^*$.
c) Prove: If S is a language over A and $A \subset S$, then $A^* = S^*$.

EXERCISE 66 Let $A = \{1, 2, 3, +, *, -\}$ ($*$ is multiplication). How many strings of length 3 are in A^*? How many make sense in standard Algebra? (For example, 12+ does not make sense, but 1+2 and 121 do.)

Challenge Problem 2: Let A and B be non-empty alphabets. Prove: $A^*B^* = B^*A^*$ if and only if $A \subset B$ or $B \subset A$.

Challenge Problem 3: Prove: for any two alphabets A and B, $A^*B^* \cap B^*A^* = A^* \cup B^*$.

Solutions for Selected String Exercises

Challenge problem 1 Let A be an alphabet and S be non-empty a set of strings over A. Prove: $S \subset S^2$ if and only if $\lambda \in S$.
$(\rightarrow)$: (Proof by contrapositive) Suppose $\lambda \notin S$. Since $S \neq \emptyset$, S must contain at least one element. Let x be an element of minimal length in S. That is, no element of S is "shorter" than x. Then for all $u \in S$, $|u| \geq |x|$. Now consider the lengths of the elements of S^2. If $u, v \in S$, then
$|uv| = |u| + |v| \geq 2|x| > |x|$ (this last inequality is because $\lambda \notin S$, so that $|x| \geq 1$). Thus $x \notin S^2$, since it is "too short". Thus there is an element that is in S, but not in S^2, so $S \not\subset S^2$.
$(\leftarrow)$: Assume $\lambda \in S$ and let u be an arb. elt. of S. Then $u = \lambda u \in S^2$. Thus every elt. of S is in S^2, so $S \subset S^2$.

Exercise 60. a) Suppose $A = \{a, b\}$ and $S = \{a, b, ab\}$. What is S^2?
Answer: $S^2 = \left\{a^2, ab, a^2b, ba, b^2, bab, aba, ab^2, (ab)^2\right\}$
b) Find an element that belongs to $S^3 \cap S^4$.
Answer: $a^3b = (a)(a)(ab) = (a)(a)(a)(b)$, so is on both S^3 and S^4.

Exercise 61
a) All but ac, ca, abc, a^2c, since an element of $A^* \cup B^*$ must be in A^* or B^* and thus cannot contain both a and c.
b) All, since $A \cup B = \{a, b, c\}$ and all are strings over $\{a, b, c\}$.
c) All but ca. (For example, $c^2b \in A^*B^*$ because $c^2b = (\lambda)(c^2b)$, where $\lambda \in A^*$ and $c^2b \in B^*$.)
d) All but ac, abc and a^2c belong to B^*A^*.

Exercise 62. For any alphabet A, what is $(A^*)^2$? How does it compare with $(A^2)^*$?
Solution: Note that every element of $(A^*)^2$ is of the form vw, where v and w are in A^*,; i.e., they are strings over A. But the concatenation of two words over A is another word over A; i.e., it is in A^*. Thus every element of $(A^*)^2$ is also in A^*. So $(A^*)^2 \subset A^*$. Also, $A^* \subset (A^*)^2$ since $\lambda \in A^*$. Thus, by Double Subset, $(A^*)^2 = A^*$.

Exercise 63. For any two alphabets A and B, prove that $A^*B^* \cap B^*A^*$ contains at least one element other than λ.
Solution: Let a be an element in one of the alphabets. Then the string a belongs to both A^*B^* and B^*A^* (since $a = a\lambda = \lambda a$). In fact, all powers of a are also in $A^*B^* \cap B^*A^*$. Thus $A^*B^* \cap B^*A^*$ contains $\{\lambda, a, a^2, a^3, \cdots\}$ and so is $\neq \{\lambda\}$.

Exercise 64. a) Let $A = \{a, b\}$ and $B = \{b, c\}$. Prove by example that $A^* \cup B^* \neq (A \cup B)^*$
Solution We must find an element that is in one set but not the other. Note that ac is in $(A \cup B)^*$, since $A \cup B = \{a, b, c\}$, but $ac \notin A^* \cup B^*$, since it is in neither A^* nor B^*.

b) Use a standard subset proof to show: for any two alphabets A and B, $A^* \cup B^* \subset (A \cup B)^*$.
Solution Let v be an arb. elt. of $A^* \cup B^*$. Then $v \in A^*$ or $v \in B^*$.
Case 1. Let $v \in A^*$. Then v is a string over A. Since $A \subset A \cup B$, v is also a string over $A \cup B$. That is, $v \in (A \cup B)^*$.

Case 2. Let $v \in B^*$. Then v is a string over B. Since $B \subset A \cup B$, v is also a string over $A \cup B$. That is, $v \in (A \cup B)^*$.
Thus every elt. of $A^* \cup B^*$ is also in $(A \cup B)^*$; so $A^* \cup B^* \subset (A \cup B)^*$.

Exercise 65. a) Prove: If S is a language over A, then $S^* \subset A^*$.
Solution:

b) Prove: if S and T are languages such that $S \subset T$, then $S^* \subset T^*$.
Solution:

c) Prove: If S is a language over A and $A \subset S$, then $A^* = S^*$.
Solution:

Exercise 66. Let $A = \{1, 2, 3, +, *, -\}$ ($+$ is addition, $-$ is subtraction and $*$ is multiplication). How many strings of length 3 are in A^*? How many are strings in the language of standard Algebra? (For example, 12+ does not make sense, but 1+2 and 121 do.)
Solution: Since A has 6 different elements (3 digits and 3 symbols), there are $6^3 = 216$ strings of length 3 over A. To determine how many make sense in standard algebra, consider the types of strings of length 3:

type 1: containing 3 digits (111, etc.)
type 2: containing 2 digits and one symbol (11+, etc.)
type 3: containing 1 digit and 2 symbols (1++, etc.)
type 4: containing 3 symbols (+++, etc.)

Clearly, no word of types 3 or 4 make sense as (stand-alone) algebraic expressions.

How many of type 1? 3 · 3 · 3 = 27. (Pick a digit, pick a digit, pick a digit.) All make sense in standard algebra.
There are 3 · 27 = 81 words of type 2, but only the ones of the form (digit)(symbol)(digit) make sense. There are 27 of these. So the total is 63.
(Note: if you consider the "$-$" as a negative sign, there are 9 more, since the form ($-$)(digit)(digit) makes sense.)

Review for Test 3

Still More Hints on Taking Tests

1. Concentration is a key factor in test performance. Fortunately, concentration is a mental skill that can be developed through practice. Many simple calculation mistakes are due to lack of concentration.

2. Your mind can process information much faster than you can write. When you are doing your homework, discipline your mind to focus on exactly what you are writing. Then bring that mental discipline with you to the test.

3. When you take a test, try to put all distractions out of your mind. Even subconsciously, you can be thinking about something that is bothering you and make errors on problems you know how to do.

4. If you do not understand what is to be done for any given problem, ask the instructor for an explanation. Often the instructor can help you clear up the misunderstanding. If not, then interpret the problem to the best of your ability, explain how you understood it, and solve it that way.

SAMPLE TEST 3 (Solutions are given in the last section of the Companion, but try not to look until you have tried the problem.)

1. (20 points) Give a demonstration using **NO THEOREMS AS REASONS** for:
$\vdash \overline{A} \subset A \cup B \ \rightarrow \ \overline{A} \subset B$

2. (20) Using your demonstration above as a guide, give an informal proof of the statement:
if $\overline{A} \subset A \cup B$, then $\overline{A} \subset B$

3. (15) Show $\overline{A \cup \overline{B}} \cup \overline{A \cup B} \ = \overline{A}$ using a "chain of equations".

4. (15) Demonstrate: $\vdash (A \cup B \subset \overline{C}) \ \rightarrow \ A \cap C = \emptyset$ You may use any Theorem, Axiom or Definition we have discussed in your demonstration.

5. (10) Fill in the missing reasons for steps 2, 3, and 4:

1.	$a \in \overline{A} \cup \overline{B}$	as.
2.	$a \in \overline{A \cap B}$	
3.	$a \notin A \cap B$	
4.	$\sim (a \in A \wedge a \in B)$	

6. [5,5,6,6] For all parts below, use the alphabets $A = \{a, b\}$ and $B = \{b, c\}$.
a) List 5 elements of A^* of length 5.
b) List the elements of AB^2.
c) Prove by example that $A^*B^* \neq B^*A^*$
d) Use a double subset argument to prove: $A^* \cap B^* = (A \cap B)^*$

Chapter 5

Templates for Chapter 5

Template 1
Given: $F = \{(x,y) : \cdots\}$
To prove: F is a **function**

Write:
Proof: Let x, y, z be arb. elts. s. t. $(x,y) \in F$ and $(x,z) \in F$.
Then $\cdots$
Thus, $y = z$. Since x, y, z were arb., F is a function. □

Template 2
Given: $F = \{(x,y) : \cdots\}$
To prove: F is a **not a function**

Do:
Find (by inspection, algebra, etc.) actual elements x, y, z (usually numbers) s.t. $y \neq z$, but $(x,y) \in F$ and $(x,z) \in F$

Write:
Proof: Observe that $(x,y) \in F$ and $(x,z) \in F$.
$\cdots$ (verification if necessary)
Since $y \neq z$, F is not a function. □

Template 3
Given: *Domain/Rule presentation* for F and G (discussed in class):
That is, we are given D(F), D(G) and rules (usually equations) for $F(x)$ and $G(x)$.
To prove: F and G are **equal**

Write: Observe D$(F) = D(G)$ [usually given - verify if necessary] and let D = D$(F) =$ D(G). Suppose x is an arb. elt. of D.
$\cdots$
Thus $F(x) = G(x)$ for every x in D, so $F = G$. □

Notation: $F : A \to B$ means F is a function with domain A and range a subset of B. Thus there are three things required for (and implied by) the use of this notation:
1) F is a function
2) D$(F) = A$
3) R$(F) \subset B$

Template 4a
Given: $F : A \to B$ and a rule for $F(x)$.
To prove: F is **onto** (notation: $F : A \underset{\text{onto}}{\to} B$ - read "F maps A onto B")
Note: since we are given $F : A \to B$, we need only prove $B \subset \text{R}(F)$.

Write: Let y be an arb. elt. of B.
. . .
Thus there is an $x \in A$ s.t. $F(x) = y$, so $y \in \text{R}(F)$. Since y was arb., $F : A \underset{\text{onto}}{\to} B$.

Template 4b
Given: a function F with domain A and a rule for $F(x)$.
To prove: $F : A \underset{\text{onto}}{\to} B$
Note: since the Domain/Rule presentation is given for F, we need to prove that $\text{R}(F) = B$.

Write: $\subset$: Let y be an arb. elt. of $\text{R}(F)$.
. . .
Then $y \in B$. Since y was arb., $\text{R}(F) \subset B$.
$\supset$: Let y be an arb. elt. of B.
. . .
Thus there is an $x \in A$ s.t. $F(x) = y$, so $y \in \text{R}(F)$. Since y was arb., $B \subset \text{R}(F)$.
Thus, $F : A \underset{\text{onto}}{\to} B$.

Template 5
Given: a function $F : A \to B$ and a rule for $F(x)$
To prove: F is **not onto**

Do: Find an actual element $y_0 \in B$ s.t. $F(x) \neq y_0$ for all $x \in A$.

Write: Let $y = y_0$. Then $y \in B$.
. . .
Thus there is no $x \in A$ s.t. $F(x) = y$, so $F : A \to B$ is not onto.

Template 6
Given: a function $F : A \to B$ and a rule for $F(x)$
To prove : F is **one-to-one (1-1)** (notation: $F : A \overset{\text{1-1}}{\to} B$)
Note: this template also works with the Domain/Rule presentation.

Write: Let x_1, x_2 be arb. elts. of A s.t. $F(x_1) = F(x_2)$.
. . .
Then $x_1 = x_2$. Since x_1 and x_2 were arb., F is 1-1.

Template 7
Given: a function $F : A \to B$ and a rule for $F(x)$
To prove : F is **not one-to-one**
Note: this template also works with the Domain/Rule presentation.

Do: find two actual elements x_1, x_2 of A s.t. $x_1 \neq x_2$ but $F(x_1) = F(x_2)$.

Write: Note that x_1 and x_2 are two different elements of A and $F(x_1) = F(x_2)$ [justify if necessary]. Thus F is not 1-1.

Question: if F were presented in set-builder form, how would you rewrite the above two templates?

Some examples and exercises for Chapter 5
Example Consider the relation F = $\left\{(x,y)\text{: } x, y \in \mathbb{R} \wedge xy \neq 0\right\}$.
Prove D(F) = $\mathbb{R}^*$ ($\mathbb{R}^*$= the set of non-0 reals)
Proof: ("Double subset" or "double inclusion" method.)
$\subset$: D(F) $\subset \mathbb{R}^*$ Let x be an arb. elt. of D(F). Then there is a y s.t. $(x, y) \in$ F. This means x and y are real and $xy \neq 0$. But this implies x is a non-0 real number; i.e. $x \in \mathbb{R}^*$. Since x was arb., D(F) $\subset \mathbb{R}^*$.
$\supset$: $\mathbb{R}^* \subset$ D(F) Let x be an arb. elt. of $\mathbb{R}^*$ and spse y is also in $\mathbb{R}^*$. Then $x, y \in \mathbb{R}$ and $xy \neq 0$, so $(x, y) \in$ F. Thus $x \in$ D(F). Since x was arb., $\mathbb{R}^* \subset$ D(F).

Example Let $F = \{(x, y) : x, y \in \mathbb{R}$ and $xy + 3y = 3\}$. Prove F is a function.
Proof: Let x, y and z be arb. elts. s.t. $(x,y) \in$ F and $(x,z) \in$ F. Then $x, y, z \in \mathbb{R}$ and $xy + 3y = 3$ and $xz + 3z = 3$. Observe that $x + 3 \neq 0$. Otherwise the equation $xy + 3y = 3$ becomes 0 = 3, a contradiction. We may thus solve for z and w: $z = \frac{3}{x+3} = w$. Since x, y, z were arb., F is a function.

EXERCISE 67.
a) Prove $F = \{(x, y) : x, y \in \mathbb{R}$ and $2y - xy = x\}$ is a function.
b) Find D(F) and R(F) for $F = \{(x, y) : x, y \in \mathbb{R}$ and $xy = 0\}$.
c) Prove: for $F = \{(x, y) : x, y \in \mathbb{R}$ and $xy + 3y = 3\}$, R(F) $= \mathbb{R}^*$.
d) Find D(F) and R(F) for $F = \{((x,y),z)\text{: } x,y,z \in \mathbb{R} \wedge xy + yz = 1\}$.

Example Let F = $\left\{((x,y),z)\text{: } x,y,z \in \mathbb{R} \wedge z\sqrt{x} = 1 + z\sqrt{y}\right\}$.
a) Prove F is a function.
Proof: Let x, y, z and w be arb. elts. s.t. $((x,y),z) \in$ F and $((x,y),w) \in$ F. Then x, y, z and $w \in \mathbb{R}$ and $z\sqrt{x} = 1 + z\sqrt{y}$ and $w\sqrt{x} = 1 + w\sqrt{y}$. Observe that $\sqrt{x} \neq \sqrt{y}$. Otherwise the equation $z\sqrt{x} = 1 + z\sqrt{y}$ becomes 0 = 1, a contradiction. We may thus solve for z and w:
$z = \frac{1}{\sqrt{x} - \sqrt{y}} = w$. Since x, y, z and w were arb., F is a function.

b) Prove $z = 2$ and $z = -2$ both belong to R(F).
Proof: Observe that $((\frac{1}{4},0),2) \in$ F and $((0,\frac{1}{4}),-2) \in$ F. This shows $2 \in$ R(F) and $-2 \in$ R(F).

Comment: This would be your entire response to the question. This can be done by example since we're actually proving an *existential statement*. To find numbers x and y s.t. $((x,y),2) \in$ F, look at the algebra in the proof above. We need to solve the equation $2 = \frac{1}{\sqrt{x}-\sqrt{y}}$. By eye, we could let $y = 0$ and then solve $2 = \frac{1}{\sqrt{x}}$ for x, getting $x = \frac{1}{4}$. That is, $((\frac{1}{4},0),2) \in$ F. Similarly, $((0,\frac{1}{4}),-2) \in$ F.

- Consider the relation G = $\left\{\left((x,y),z\right): x,y,z \in \mathbb{R} \text{ and } 3(x^2-1)+zx = 9+zy\right\}$.Prove G is not a function.

Proof: Observe that, for example, $((2,2),0) \in$ G and $((2,2),1) \in$ G and $0 \neq 1$. Thus G is not a function.

Comment: this is another existential statement, and can thus be proved by example. We need to find two pairs $((x,y),z) \in$ G and $((x,y),w) \in$ G, with $z \neq w$. Unless you're clever enough to "observe" a solution (few are), the best way to find such pairs is to pretend we're trying to solve for z in the defining equation for G: $3(x^2-1)+zx = 9+zy$. We would get $z = \frac{9-3(x^2-1)}{x-y}$. Note that, if this were "legal" in G, then G would be a function. So there must be a problem here, and it must be with a 0 denominator $(x-y=0)$. So, spse $x = y$. Then the equation $3(x^2-1)+zx = 9+zy$ becomes $3(x^2-1) = 9$, yielding $x^2 = 4$. z dropped out and can thus be chosen arbitrarily. If we take $x = 2$, we'll get the solution above. Note for test purposes, the one line is sufficient for an answer to this type of question. The work would be done on scratch paper (or in your head if you're that clever).

Example Suppose F and G are functions such that D(F) = D(G) = $\mathbb{R}$ and

$i)$ $\forall x \in \mathbb{R}$ $\quad F(x) = \frac{|x|+5x}{2}$

$ii)$ $G(x) = \begin{cases} \_\_\_\_\_\_ & \text{if } x \geq 0 \\ \_\_\_\_\_\_ & \text{if } x < 0 \end{cases}$.

a) Fill in the blanks in the definition of G so that F = G.
Note, if $x \geq 0$, $|x| = x$, so $F(x) = \frac{x+5x}{2} = 3x$; and if $x < 0$, $|x| = -x$, so $F(x) = \frac{-x+5x}{2} = 2x$. Thus, $3x$ and $2x$ are the values in the blanks of G.

b) Prove that F = G.
Proof: First, note that D(F) = D(G), as given in the statement of the problem. We must show that $F(x) = G(x)$ for every x in the common domain ($\mathbb{R}$). So, let x be an arb. elt. of $\mathbb{R}$.
Case 1: spse $x \geq 0$. Then $|x| = x$, so $F(x) = \frac{x+5x}{2} = 3x = G(x)$ (as determined in part a)). Thus $F(x) = G(x)$ in case 1.
Case 2: spse $x < 0$. Then $|x| = -x$, so $F(x) = \frac{-x+5x}{2} = 2x = G(x)$ (again, as determined above). Thus $F(x) = G(x)$ in case 2.
Since $F(x) = G(x)$ in all cases, F = G.

Example For parts a) and b) below, suppose F and G are relations (sets of ordered pairs) such that F $\cup$ G is a function.

a) Prove F and G are both functions

Proof: Let's prove F is function. Let x, y and z be arb. elts. s.t. (x, y) and $(x, z) \in$ F. Since F $\subset$ F $\cup$ G, (x, y) and $(x, z) \in$ F $\cup$ G. But since F $\cup$ G is a function, $y = z$. Since x, y and z were arb., F is s function. The proof for G is similar.

b) Prove $\forall x \in$ D(F) $\cap$ D(G) [F(x) = G(x)].

Proof: Let x be an arb. elt. of D(F) $\cap$ D(G). Then $x \in$ D(F). Thus there is a y s.t. $(x, y) \in$ F. Then $y =$ F(x). Similarly there is a z s.t. $(x, z) \in$ G and $z =$ G(x). But since both F and G are subsets of F $\cup$ G, (x, y) and $(x, z) \in$ F $\cup$ G. Since F $\cup$ G is a function, $y = z$.

EXERCISE 68. Suppose F and G are functions such that D(F) = D(G) = $\mathbb{R}^*$ and

$i)$ $\forall x \in \mathbb{R}^*$ F(x) = $\frac{|2x| + x}{|x|}$

$ii)$ G(x) = $\begin{cases} 3 \text{ if } x > 0 \\ 1 \text{ if } x < 0 \end{cases}$.

Prove F = G.

EXERCISE 69. In each case, tell whether the given set is a function (Y) or not a function (N). No proofs required.

______a) $\{(x,y): x,y \in \mathbb{N} \wedge x < 2 \wedge y > 2\}$

______b) $\{(x,y): x,y \in \mathbb{R} \wedge y + 2 = yx + x^2\}$

______c) $\{(x,y): x,y \in \mathbb{R} \wedge y + 1 = yx + x^2\}$

______d) $\{(x,y): x,y \in \mathbb{R} \wedge x^2 = 1 \wedge y = 4\}$

______e) $\{(x,y): x,y \in \mathbb{R} \wedge x = 1 \wedge y^2 = 4\}$

- Let $A = [0, \infty)$. Suppose D$(F) = A$ and $\forall x \in A$ $F(x) = x^2 + 1$. Prove: $F : A \overset{\text{1-1}}{\to} A$, but $F : A \underset{\text{onto}}{\to} A$ is false.

Proof: To prove F is 1-1, we use Option 1 from Template 6 above. Let x_1 and x_2 be arbitrary elements of A s.t. $F(x_1) = F(x_2)$. Then $x_1^2 + 1 = x_2^2 + 1$, so $x_1^2 = x_2^2$. This implies that $x_1 = \pm x_2$. But since x_1 and $x_2 \geq 0$, this says $x_1 = x_2$. Since x_1 and x_2 were arb., F is 1-1. To prove F is not onto, we use Template 5. Let $y = 0$. Then $y \in A$. Suppose there is an x in A s.t. $F(x) = 0$. Then $x^2 + 1 = 0$, so $x^2 = -1$, which is a contradiction. Thus there is no $x \in A$ s.t. $F(x) = y$, so $F : A \to A$ is not onto.

Chapter 6

Note: throughout Chapter 6, you may assume the variable n represents an integer that is ≥ 0.

Template 1. Proof by Mathematical Induction (PMI)
Given : a sequence of statements P(1), P(2), P(3), $\cdots$
To prove: P(n) is true for all $n \geq 1$

Write:
Let P(n) be the statement [write out P(n)].
Base step - P(1) : [write out P(1)]. True by [reason].
Inductive hypothesis - Assume P(k) : [write out P(k)]
Inductive step - Prove P$(k+1)$: [write out P$(k+1)$]
$\cdots$
Thus P$(k+1)$ is true.
By PMI, P(n) is true for all $n \geq 1$.

Some PMI examples.
• For all natural numbers n, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
Proof: Let P(n) be the statement $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
Base - P(1): 1 = $\frac{1(1+1)}{2}$. True.
Ind. hyp. - Assume P(k) : $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.
Ind. step - Prove P$(k+1)$: LHS = $1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}$ = RHS

$$
\begin{aligned}
\text{LHS} = 1 + 2 + \cdots + (k+1) &= 1 + 2 + \cdots + k + (k+1) \\
&= \frac{k(k+1)}{2} + k + 1 \text{ (by ind. hyp.)} \\
&= \frac{k(k+1)}{2} + \frac{2k+2}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2} = \text{RHS}
\end{aligned}
$$

Thus P$(k+1)$ is true. By PMI, P(n) is true for all natural numbers n.□

• For all $n \geq 1$, $11^n - 1$ is a multiple of 10.
Proof: Let P(n) be the statement $11^n - 1$ is a multiple of 10.
Base - P(1): $11^1 - 1$ (= 10) is a multiple of 10. True.
Ind. hyp. - Assume P(k): $11^k - 1$ is a multiple of 10 (i.e., $11^k - 1 = 10j$, where $j \in \mathbb{I}$).

Ind. step - Prove P$(k+1)$: $11^{k+1} - 1$ is a multiple of 10.

$$
\begin{aligned}
11^{k+1} - 1 &= 11 \cdot 11^k - 1 \\
&= 11(10j+1) - 1 \text{ (by induction hypothesis)} \\
&= 11 \cdot 10j + 11 - 1 \\
&= 10(11j+1), \text{ which is a multiple of 10.}
\end{aligned}
$$

Thus P$(k+1)$ is true. By PMI, P(n) is true for all $n \geq 1$.□

Example Using Theorem 4.18 ($\overline{A \cap B} = \overline{A} \cup \overline{B}$), we will prove:
For all $n \geq 2$, $\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}$.
Proof: Let P(n) be the statement $\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}$.
Base - P(2): $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$ True by T4.18
Ind. hyp. - Assume P(k): $\overline{A_1 \cap A_2 \cap \cdots \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k}$
Ind. step - Prove P($k+1$): LHS=$\overline{A_1 \cap A_2 \cap \cdots \cap A_{k+1}} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k} \cup \overline{A_{k+1}}$= RHS
LHS = $\overline{A_1 \cap A_2 \cap \cdots \cap A_{k+1}} = \overline{A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}}$
$= \overline{A_1 \cap A_2 \cap \cdots \cap A_k} \cup \overline{A_{k+1}}$ (by T4.18)
$= \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k} \cup \overline{A_{k+1}}$ (by Ind. hyp.)
= RHS
Thus P($k+1$) is true. By PMI, P(n) is true for $n \geq 2$.☐

(a)EXERCISE 70 Prove by PMI: for all $n \geq 0$, $\frac{(2n)!}{2^n n!}$ is an integer. (Note: $0! = 1$.)

Example Prove by PMI: For $n = 2, 3, 4, \cdots$, $(2n+1)! > 2^{2n}(n!)^2$ (to be done in class)

(t)EXERCISE 71 Prove by PMI: For all $n \geq 1$, $1 + 4 + 7 + \cdots + (3n+1) = \frac{(n+1)(3n+2)}{2}$

The Fibonacci Numbers F_n are the positive integers in the sequence defined as follows:

$$\begin{cases} F_1 = 1, F_2 = 1 \\ F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 2 \ \ (\textit{Fibonacci recursion formula}) \end{cases}$$

The first 10 Fibonacci numbers are: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

Example Let's prove: for all $n \geq 2$, $F_nF_{n+1} - F_nF_{n-1} = F_n^2$. Suppose $n \geq 2$. Then $F_nF_{n+1} - F_nF_{n-1} = F_n(F_n + F_{n-1}) - F_nF_{n-1} = F_n^2 + F_nF_{n-1} - F_nF_{n-1} = F_n^2$.
(Question: do you see why we needed the assumption $n \geq 2$?)

(a)EXERCISE 72 Prove that, if F_n is even, then $F_{n+1}^2 - F_{n-1}^2$ is a multiple of 4. This problem does not require induction. (Hint: factor and use the Fibonacci recursion formula.).

Let's do an induction proof with the Fibonacci Numbers.
Example Prove that, for $n \geq 1$, $F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}$.
Proof: Let P(n) be the statement $F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}$.
Base - P(1): $F_1^2 = F_1F_2$. True, since $1^2 = 1 \cdot 1$.
Ind. Hyp. - Assume P(k): $F_1^2 + F_2^2 + \cdots + F_k^2 = F_kF_{k+1}$.
Ind. Step - Prove P($k+1$): $F_1^2 + F_2^2 + \cdots + F_{k+1}^2 = F_{k+1}F_{k+2}$. (LHS=RHS)
Proof of ind. step: LHS = $F_1^2 + F_2^2 + \cdots + F_{k+1}^2 = F_1^2 + F_2^2 + \cdots + F_k^2 + F_{k+1}^2$
$= F_kF_{k+1} + F_{k+1}^2$ (by inductive hypothesis)
$= F_{k+1}(F_k + F_{k+1})$
$= F_{k+1}F_{k+2}$ = RHS
Thus P($k+1$) is true. By PMI, P(n) is true for all $n \geq 1$.☐

(a)EXERCISE 73 Prove by PMI: for all integers $n \geq 1$, $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

Example Prove that $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$ for all $n \geq 2$.
Proof: Let $P(n)$ be the statement $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$.
Base - $P(2)$: $F_2^2 - F_{2+1}F_{2-1} = 1^2 - 2 \cdot 1 = -1 = (-1)^{2+1}$.
Ind. Hyp. - Assume $P(k)$: $F_k^2 - F_{k+1}F_{k-1} = (-1)^{k+1}$
Ind. Step - Prove $P(k+1)$: $F_{k+1}^2 - F_{k+2}F_k = (-1)^{k+2}$. (LHS=RHS)
Proof of ind. step: LHS = $F_{k+1}^2 - F_{k+2}F_k = F_{k+1}^2 - (F_{k+1} + F_k)F_k$

$$= F_{k+1}^2 - F_{k+1}F_k - F_k^2$$
$$= F_{k+1}(F_{k+1} - F_k) - F_k^2$$
$$= F_{k+1}F_{k-1} - F_k^2$$
$$= -(F_k^2 - F_{k+1}F_{k-1})$$
$$= -(-1)^{k+1} \quad \text{(by ind. hyp.)}$$
$$= (-1)^{k+2} = \text{RHS}$$

Thus $P(k+1)$ is true. By PMI, $P(n)$ is true for all $n \geq 2$.

(a)EXERCISE 74 Prove by PMI: $\forall n \in \mathbb{N}$, $F_1 + 2F_2 + \ldots + nF_n = (n+1)F_{n+2} - F_{n+4} + 2$.

(a)EXERCISE 75 Prove by PMI that $F_n \leq 2^n$ for all $n \geq 1$. (Hint: $F_{k+1} = F_k + F_{k-1} < 2F_k$.)

(a)EXERCISE 76 Let $a_1 = 2$ and $a_{n+1} = 2a_n + 1$ for $n \geq 1$. Use PMI to prove $a_n < 2^{n+1}$ for all $n \geq 1$. (Hint: since each a_n is an integer, $a_n < 2^{n+1}$ if and only if $a_n \leq 2^{n+1} - 1$.)

(a)EXERCISE 77 Let $a_0 = 1$ and $a_{n+1} = a_n + \left(\frac{1}{2}\right)^{n+1}$ for all $n \geq 0$. Prove by PMI that $a_n = 2 - \left(\frac{1}{2}\right)^n$ for all $n \geq 0$.

Strong Induction (The Second Principle of Mathematical Induction)
Suppose $S \subset \mathbb{N}$ and
(a) $1 \in S$, and
(b) whenever $1, 2, \cdots, k \in S$, then $k+1 \in S$.
Then $S = \mathbb{N}$.

Template 2. Strong Induction (Second PMI)
Given: a sequence of statements $P(1)$, $P(2)$, $P(3)$, $\cdots$
To prove: $P(n)$ is true for all $n \geq 1$

Write:
Let $P(n)$ be the statement [write out $P(n)$].
Base step - $P(1)$: [write out $P(1)$]. True by [reason].
Inductive hypothesis - Assume $P(1), \cdots, P(k)$: [write out $P(1), \cdots, P(k)$]
Inductive step - Prove $P(k+1)$: [write out $P(k+1)$]
$\cdots$
Thus $P(k+1)$ is true.
By PMI, $P(n)$ is true for all $n \geq 1$.

Here is one of the classic Strong Induction proofs:

Theorem For every integer $n \geq 2$, n is either prime or a product of primes.
(Note: a *prime* number is an integer ≥ 2 which has only 1 and itself as factors.)
Proof: Let P(n) be the statement "n is either a prime or a product of primes."
Base - P(2): 2 is either a prime or a product of primes - true since 2 is a prime.
Ind. Hyp. - Assume P(2) $\wedge$ P(3) $\wedge$ . . . $\wedge$ P(k): that is, suppose every number from 2 to k is either prime or a product of primes.
Ind. Step - Prove P($k+1$): $k+1$ is either prime or a product of primes.
If $k+1$ is prime, this is certainly true. If $k+1$ is not prime, then it can be factored: $k+1 = ab$, where $2 \leq a,b \leq k$. By the induction hypothesis, each of a and b is either prime or a product of primes. Putting this together, $k+1$ is a product of primes. Thus P($k+1$) is true.
By Second PMI, P(n) is true for all $n \geq 2$. ☐

An important variation on Strong Induction is PMIv3 (the "**Fibonacci Variation**").

Template 3. PMI, variation 3 (PMIv3, or "Fibonacci Variation")
Given: a sequence of statements P(1), P(2), P(3), $\cdots$
To prove: P(n) is true for all $n \geq 1$

Write:
Let P(n) be the statement [write out P(n)].
Base step - P(1) : [write out P(1)]. True by [reason].
<u>and</u> P(2) : [write out P(2)]. True by [reason].
Inductive hypothesis - Assume P(k) and P$(k+1)$ [write out P(k) and P$(k+1)$].
Inductive step - Prove P$(k+2)$: [write out P$(k+2)$]
Proof of ind. step: $\cdots$
Thus P$(k+2)$ is true.
By PMIv3, P(n) is true for all $n \geq 1$.

Example Use the Fibonacci Variation to prove: $\forall n \in \mathbb{N}$, $3\text{F}_{n+2} - \text{F}_n = \text{F}_{n+4}$.
Proof: Let P(n) be the statement $3\text{F}_{n+2} - \text{F}_n = \text{F}_{n+4}$.
Base - P(1): $3\text{F}_3 - \text{F}_1 = \text{F}_5$. Correct, since $3 \cdot 2 - 1 = 5$.
<u>and</u> P(2): $3\text{F}_4 - \text{F}_2 = \text{F}_6$. Again correct, since $3 \cdot 3 - 1 = 8$.
Ind. hyp. - Assume P(k) and P($k+1$): that is assume $3\text{F}_{k+2} - \text{F}_k = \text{F}_{k+4}$ and $3\text{F}_{k+3} - \text{F}_{k+1} = \text{F}_{k+5}$.
Ind. Step - prove P($k+2$): $3\text{F}_{k+4} - \text{F}_{k+2} = \text{F}_{k+6}$.
Proof of ind. step: LHS = $3\text{F}_{k+4} - \text{F}_{k+2} = 3(\text{F}_{k+3} + \text{F}_{k+2}) - (\text{F}_{k+1} + \text{F}_k)$ (by the Fibonacci recursion formula)

$= 3\text{F}_{k+3} - \text{F}_{k+1} + 3\text{F}_{k+2} - \text{F}_k$
$= \text{F}_{k+5} + \text{F}_{k+4}$ (by inductive hypothesis)
$= \text{F}_{k+6}$ = RHS

Thus P$(k+2)$ is true. By the PMIv3, P(n) is true for all $n \in \mathbb{N}$.☐
Do you see why we had to check *both* P(1) and P(2) at the beginning of this proof? And why we assumed $k \geq 2$ in the Inductive Hypothesis? We'll discuss this in class.

Let's do another proof using the Fibonacci Variation of PMI.

Example Given: $a_0 = 2,\ a_1 = 2$ and $a_{n+1} = 2a_n + 3a_{n-1}$ for $n \geq 1$. Prove by PMIv3: for all $n \geq 0,\ a_n = 3^n + (-1)^n$.

Let P(n) be the statement $a_n = 3^n + (-1)^n$.
Base - P(0) : $a_0 = 3^0 + (-1)^0$; i.e., $2 = 1 + 1$. True by given info.
P(1) : $a_1 = 3^1 + (-1)^1$; i.e., $2 = 3 - 1$. True by given info.
Inductive hypothesis - Assume P(k) and P$(k+1)$: $a_k = 3^k + (-1)^k$ and $a_{k+1} = 3^{k+1} + (-1)^{k+1}$.
Inductive step - Prove P$(k+2)$: LHS $= a_{k+2} = 3^{k+2} + (-1)^{k+2} =$ RHS
Proof: LHS $=$

$$\begin{aligned} a_{k+2} &= 2a_{k+1} + 3a_k \text{ (by given info)} \\ &= 2\left(3^{k+1} + (-1)^{k+1}\right) + 3\left(3^k + (-1)^k\right) \text{ (by ind. hyp.)} \\ &= 2 \cdot 3^{k+1} + 3^{k+1} + 2(-1)^{k+1} + 3(-1)^k \\ &= 3 \cdot 3^{k+1} - 2(-1)(-1)^{k+1} + 3(-1)^k(-1)^2 \\ &= 3^{k+2} + (-1)^{k+2} \\ &= \text{RHS.} \end{aligned}$$

Thus P$(k+2)$ is true. By PMIv3, P(n) is true for all $n \geq 0$.

EXERCISE 78 Use the Fibonacci Variation to prove: $F_{n+5} = 5F_{n+2} - 2F_n$ for all $n \geq 1$.

This challenge exercise is a generalization of the above ideas.
(*)Challenge Suppose c is an integer and $c \geq 3$.Use the Fibonacci Variation to prove: for all $n \geq 1,\ F_{n+c} = F_c F_{n+2} - F_{c-2} F_n$.

EXERCISE 79 Prove by PMIv3 that $F_n \leq 2^n$ for all $n \geq 1$.

EXERCISE 80 A sequence a_n is defined: $a_1 = 1,\ a_2 = 2,\ a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 1$. Use the Fibonacci Variation to prove that $a_n = 2^{n-1}$ for all $n \geq 1$.

Additional PMI Exercises

(w)EXERCISE 81 Prove by PMI: for $n \geq 1$, $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

(w)EXERCISE 82 Prove by PMI: for $n \geq 1,\ 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

(w)EXERCISE 83 Prove by PMI: for $n \geq 1$, $n^3 - n$ is a multiple of 3.

(w)EXERCISE 84 Prove by PMI: for $n \geq 1,\ 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$

(w)EXERCISE 85 Prove by PMI: for $n \geq 1$, $4^{2n} - 1$ is a multiple of 15.

Example Prove by PMI: $2^n > n^2$ for $n \geq 5$.
Proof: Let P(n) be the statement $2^n > n^2$.
Base - P(5): $2^5 > 5^2$. True, since $32 > 25$..
Ind. hyp. - Assume P(k) : $2^k > k^2$.
Ind. step - Prove $\mathrm{P}(k+1) : 2^{k+1} > (k+1)^2$ (LHS > RHS).
Proof of ind. step: LHS $= 2^{k+1} = 2 \cdot 2^k > 2k^2 = k^2 + k^2$
$\geq k^2 + 5k$ (since $k \geq 5$)
$\geq k^2 + 2k + 1$
$= (k+1)^2 =$ RHS.
Thus P($k+1$) is true. By PMI, P(n) is true for $n = 5,\ 6,\ 7,\ \cdots$.□

(w)EXERCISE 86 Prove by PMI: for $n \geq 1$, $2^{2n} < (n+2)!$

EXERCISE 87 Prove by PMI: $\forall n \geq 4$, $n! > 2^n$

EXERCISE 88 Prove by PMI: for $n \geq 1$, $(n!)^2 < (2n)!$

(w)EXERCISE 89 Recall Theorem 4.24: $\vdash$ A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C).
Prove the following generalization by PMI:
For all $n \geq 2$, $A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$.

EXERCISE 90 Prove by PMI: for every natural number n, the set {1,2,3, $\cdots$,n} contains exactly $n(n-1)/2$ subsets of size 2. (For example, the set {1,2,3} has these subsets of size 2: {1,2}, {1,3} and {2,3}; i.e., it has 3(3-1)/2 = 3 subsets of size 2.)

(w)EXERCISE 91 Prove that the number of subsets of $\{1, 2, 3, \cdots, n\}$ of size 3 is $\frac{n(n-1)(n-2)}{6}$.

EXERCISE 92 A sequence a_n is defined: $a_1 = 5,\ a_{n+1} = a_n + n + 5$ for $n \geq 1$. Use PMI to prove that $a_n = \frac{n(n+9)}{2}$ for all $n \geq 1$.

EXERCISE 93 A sequence a_n is defined: $a_1 = 3,\ a_{n+1} = 2a_n + 1$. Use PMI to prove that $a_n = 2^{n+1} - 1$ for all $n \geq 1$.

(w)EXERCISE 94 A sequence a_n is defined: $a_0 = 2,\ a_1 = 1,\ a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 0$. Use the Fibonacci Variation to prove that $a_n = 2^n + (-1)^n$ for all $n \geq 0$.

(w)EXERCISE 95 Prove: for $n \geq 1$, $1 + \frac{n}{2} \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \leq 1 + n$

(w)EXERCISE 96 Prove: for $n \geq 1$, $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1} n^2 = (-1)^{n+1} \frac{n(n+1)}{2}$

EXERCISE 97 Given the Theorem: If A ⊂ C and B ⊂ C, then A ∪ B ⊂ C.
Prove by PMI: for all $n \geq 2$, if $A_1 \subset C,\ A_2 \subset C,\ \cdots$, and $A_n \subset C$, then $A_1 \cup A_2 \cup \cdots \cup A_n \subset C$.

(w)Challenge Use Second PMI (strong induction) to prove that every positive integer can be expressed as a sum of (one or more) *distinct* powers of 2. (The term "distinct" means "different from each other".) For example: $13 = 2^0 + 2^2 + 2^3$, $26 = 2^1 + 2^3 + 2^4$, $16 = 2^4$

Review for Test 4/Final Exam
Studying for a Comprehensive Final Exam

1. Study the past exams and quizzes. Be sure that you know how to solve those types of problems.
2. Go over the homework problems. Concentrate on the kinds of homework problems that were emphasized in class or appeared on past exams.
3. Look at the examples that are worked out in the text or are in your notes.
4. Review any topics that were covered since the last exam.
5. You should be able to spend more time on material from the beginning of the course, and a little less on more recent material, since this will be fresher in your mind. But, do not omit any major topics.
6. In a mathematics class, you *must* work problems, more problems and still more problems.

SAMPLE FINAL EXAM (Solutions are given in the last section of the Companion, but try not to look until you have tried the problem.)

PART I - Review of Chapters 1-4

1. [20] Demonstrate $\text{P} \to \text{R},\ \text{P} \to \text{S},\ \sim \text{P} \to \text{G},\ \sim \text{G} \ \vdash\ \text{R} \wedge \text{S}$
2. [20] Demonstrate using IP: $\text{A} \to \text{B},\ \sim \text{A} \to\ \sim \text{C},\ \sim (\text{C} \wedge \text{B})\ \vdash\ \sim \text{C}$
3. [20] Demonstrate $\forall x \Big\{ \sim \text{G}(x) \vee \text{H}(x) \Big\},\ \forall x \text{G}(x) \vee \forall x \text{H}(x)\ \vdash\ \forall x \text{H}(x)$
4. [20] Demonstrate
$\sim \forall x \exists y \Big\{ \text{P}(x,y) \wedge \sim \text{Q}(x,y) \Big\},\ \forall x \exists y \Big\{ \text{P}(x,y) \wedge \text{R}(x,y) \Big\}\ \vdash \exists x \exists y \text{Q}(x,y) \wedge \exists x \exists y \text{R}(x,y)$
5. [20] Demonstrate: $\vdash A \subset \overline{A} \cup \overline{B}\ \to A \cap B = \emptyset$.
6. [3,3,3] Suppose A is the alphabet $\{a, b\}$ and S is the language $\{\lambda, a, b, ab\}$.
a) What is A^2?
b) How many elements are in S^2?
c) Prove by example: $A^* \neq (A^2)^*$

PART II - Chapters 5 and 6

NOTE: All proofs in Problems 7-12 are informal.
7. [20] Use PMI and Th. 4.24 to prove:
For all $n \geq 2,\ A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$

8. [20] **Choose** either **A** or **B** and prove by Mathematical Induction (PMI):
A. For all $n \geq 1$, $\frac{6^{2n}-1}{7}$ is an integer.
B. $\forall n \geq 1$, $(n!)^2 < (2n)!$

9. [20] Suppose F and G are functions such that D(F) = D(G) = $(-\infty, 2)$ and

i) $\forall x \in \mathbb{R}$ F(x) = $\frac{2-|x|}{x-2}$

ii) G(x) = $\begin{cases} -1 \text{ if } 0 \leq x < 2 \\ 1 + \frac{4}{x-2} \text{ if } x < 0 \end{cases}$.

Prove that F = G.

10. [20] Prove F = $\left\{\left((x,y),z\right): x,y,z \in \mathbb{R} \text{ and } -6+zx^2 = zy)\right\}$ is a function.

11. [20] Let G = $\left\{\left((x,y),z\right): x,y,z \in \mathbb{R} \text{ and } 2y+zx^2 = zy)\right\}$. Prove G is <u>not</u> a function.

12. Let $A = \mathbb{R}$ and $B = [0, \infty)$. Suppose D$(F) = A$ and $\forall x \in A$ $F(x) = |x-1|$. Prove:

b) R$(F) = B$

b) $F : A \overset{1\text{-}1}{\to} B$ is false.

Appendix: Solutions to Sample Tests

Sample Test 1 Solutions

1.a) DictionaryL: The teacher lets me.
S: I'll skip test 2.
T: I'll skip test 3.

Translation: $L \rightarrow (S \wedge T)$

b) $\sim (L \rightarrow (S \wedge T)) \leftrightarrow L \wedge \sim (S \wedge T) \leftrightarrow L \wedge (\sim S \vee \sim T)$

c) The teacher lets me, but I won't skip test 2 or I won't skip test 3. (Questionable wording: The teacher lets me, but I won't skip test 2 or test 3. This seems to say you won't skip either test.)

2.

1.	$Q \rightarrow P$	Pr.
2.	$\sim S \rightarrow \sim P$	Pr.
3.	$\sim(R \rightarrow T)$	Pr.
4.	| Q	As.
5.	| P	Det., 1,4
6.	| $P \rightarrow S$	EQ., C, 2
7.	| S	Det., 5, 6
8.	| $R \wedge \sim T$	EQ., NI, 3
9.	| R	TI, S, 8
10.	| $R \wedge S$	Rep., 7, 9
11.	$Q \rightarrow (R \wedge S)$	RCP

3.

1.	$\sim(P \wedge \sim Q)$	Pr.
2.	$(T \vee S) \rightarrow P$	Pr.
3.	T	Pr.
4.	$T \vee S$	TI, A, 3
5.	P	Det. 2,4
6.	$\sim P \vee Q$	EQ., NC, 1 (DN)
7.	Q	TI, DS, 5, 6, (DN)

4.

1.	| $\sim Q$	As.
2.	| $R \vee Q$	Pr.
3.	| $R \rightarrow P$	Pr.
4.	| $(S \vee P) \rightarrow (R \wedge Q)$	Pr.
5.	| R	TI, DS, 1,2
6.	| P	Det. 3, 5
7.	| $S \vee P$	TI, A, 6
8.	| $R \wedge Q$	Det., 4,7
9.	| Q	TI, S, 8
10.	| $Q \wedge \sim Q$	Rep., 1,9
11.	$\sim Q \rightarrow (Q \wedge \sim Q)$	RCP
12.	Q	TI, IP, 11

5. a) I b) V c) I

6.

	Statements	Reasons
1.	$\sim [(P \vee Q) \rightarrow R]$	As.
2.	$P \rightarrow R$	Pr.
3.	$Q \rightarrow R$	Pr.
4.	$(P \vee Q) \wedge \sim R$	EQ 1, NI
5.	$\sim R$	TI, S, 4
6.	$\sim P$	TI, MT, 2, 5
7.	$\sim Q$	TI, MT, 3, 5
8.	$P \vee Q$	TI, S, 4
9.	Q	TI, DS, 6, 8
10.	$Q \wedge \sim Q$	Rep., 7, 9
11.	$\sim [(P \vee Q) \rightarrow R] \rightarrow (Q \wedge \sim Q)$	RCP
12.	$(P \vee Q) \rightarrow R$	TI, IP, 11

Sample Test 2 Solutions

1. a) The Energizer Bunny does not have floppy ears.
b) Not every animal with floppy ears is a bunny.
c) $\sim F(e) \wedge \exists x[B(x) \wedge F(x)]$

2. a) Q(3,3) says: $3^2 + 3^2 \geq 10$. This is true.
b) - f) F, T, F, T, T

3.

1.	$\sim \forall x Q(x)$	Pr.
2.	$\forall x[P(x) \rightarrow Q(x)]$	Pr.
3.	$\exists x \sim Q(x)$	EQ 1, NU
4.	$\sim Q(a)$	EI, 3
5.	$P(a) \rightarrow Q(a)$	UI, 2
6.	$\sim P(a)$	TI, MT, 4, 5
7.	$\exists x \sim P(x)$	EG, 6

4.

1.	$\forall x \sim Q(x)$	Pr.
2.	$\exists x Q(x) \vee \forall x T(x)$	Pr.
3.	$\sim \exists x Q(x)$	EQ 1, NE
4.	$P(a)$	As.
5.	$\forall x T(x)$	TI, DS, 2, 3
6.	$T(a)$	UI, 5
7.	$P(a) \rightarrow T(a)$	RCP
8.	$\forall x[P(x) \rightarrow T(x)]$	UG, 7

5.

1.	$\forall x\exists y[\mathrm{F}(x,y) \to \mathrm{G}(x,y)]$	Pr.
2.	$\exists x\forall y[\mathrm{G}(x,y) \to \mathrm{H}(x,y)]$	Pr.
3.	$\forall y[\mathrm{G}(a,y) \to \mathrm{H}(a,y)]$	EI, 2.
4.	$\exists y[\mathrm{F}(a,y) \to \mathrm{G}(a,y)]$	UI, 1
5.	$\mathrm{F}(a,b) \to \mathrm{G}(a,b)$	EI, 4
6.	$\mathrm{G}(a,b) \to \mathrm{H}(a,b)$	UI, 3
7.	$\mathrm{F}(a,b) \to \mathrm{H}(a,b)$	TI, HS, 5, 6
8.	$\exists y[\mathrm{F}(a,y) \to \mathrm{H}(a,y)]$	EG, 6
9.	$\exists x\exists y[\mathrm{F}(x,y) \to \mathrm{H}(x,y)]$	EG, 8

6. Let $U = \{1,2\}$, $\mathrm{P}(x) : x = 1$, $\mathrm{Q}(x) : x = 2$ (not the only correct answer)

Sample Test 3 Solutions

1.

1.	$\overline{A} \subset A \cup B$	as.
2.	$\forall x(x \in \overline{A} \to x \in A \cup B)$	EQ., D1, 1
3.	$a \in \overline{A}$	as.
4.	$a \in \overline{A} \to a \in A \cup B$	UI, 2
5.	$a \in A \cup B$	Det., 3,4
6.	$a \in A \vee a \in B$	EQ., A5, D3, 5
7.	$a \notin A$	EQ., A5, D2, 3
8.	$a \in B$	TI, DS, 6,7
9.	$a \in \overline{A} \to a \in B$	RCP
10.	$\forall x(x \in \overline{A} \to x \in B)$	UG, 9
11.	$\overline{A} \subset B$	EQ., D1, 10
12.	$\overline{A} \subset A \cup B \to \overline{A} \subset B$	RCP

2. Proof: Assume $\overline{A} \subset A \cup B$ and let x be an arb. elt. of $\overline{A}$. Since $\overline{A} \subset A \cup B$, x belongs to $A \cup B$. That is, $x \in A$ or $x \in B$. But, since $x \in \overline{A}$, $x \notin A$. Thus $x \in B$. Therefore, every elt. of $\overline{A}$ belongs to B; so $\overline{A} \subset$ B. ☐

3. Show $\overline{A \cup \overline{B}} \cup \overline{A \cup B} = \overline{A}$ using a "chain of equations".

$$\begin{aligned}\overline{A \cup \overline{B}} \cup \overline{A \cup B} &= \overline{(A \cup \overline{B}) \cap (A \cup B)} \text{ (Th. 18)}\\ &= \overline{A \cup (\overline{B} \cap B)} \quad \text{(Th. 24)}\\ &= \overline{A \cup \emptyset} \quad \text{(Th. 13)}\\ &= \overline{A}. \quad \text{(Th. 9)}\end{aligned}$$

4.

1.	$A \cup B \subset \overline{C}$	as.
2.	$\forall x(x \in A \cup B \to x \in \overline{C})$	EQ., D1, 1
3.	$a \in A \cap C$	as.
4.	$a \in A \wedge a \in C$	EQ., A5, D4, 3
5.	$a \in A \cup B \to a \in \overline{C}$	UI, 2
6.	$a \in A$	TI, S, 4
7.	$a \in A \vee a \in B$	TI, A, 6
8.	$a \in A \cup B$	EQ., A5, D3, 7
9.	$a \in \overline{C}$	Det. 5, 8
10.	$a \notin C$	EQ., A5, D2, 9
11.	$a \in C$	TI, S, 4
12.	$a \in C \wedge a \notin C$	rep., 10, 11
13.	$a \in A \cap C \to (a \in C \wedge a \notin C)$	RCP
14.	$a \notin A \cap C$	TI, IP, 13
15.	$\forall x(x \notin A \cap C)$	UG, 14
16.	$A \cap C = \emptyset$	EQ., A4, 15
17.	$A \cup B \subset \overline{C} \to A \cap C = \emptyset$	RCP

5. Fill in the missing reasons for steps 2, 3, and 4:

1.	$a \in \overline{A} \cup \overline{B}$	as.	
2.	$a \in \overline{A \cap B}$	SEQ, Th. 18, 1	
3.	$a \notin A \cap B$	EQ., A5, D2, 2	(note: this is our "shortcut")
4.	$\sim (a \in A \wedge a \in B)$	EQ., A5, D4, 3	(")

6. a) some examples: $a^5, a^4b, a^3ba, ababa, a^2ba^2$, etc.
b) $AB^2 = \{a,b\}\{b^2, bc, cb, c^2\} = \{ab^2, abc, acb, ac^2, b^3, b^2c, bcb, bc^2\}$
c) Find an element that is in one side but not the other: note that $ac \in A^*B^*$, but $\notin B^*A^*$.
d) $\subset$: Let v be an arb elt of $A^* \cap B^*$. Then v is simultaneously a string over $\{a,b\}$ and over $\{b,c\}$. That is, the only symbol in v is c. Thus $v \in \{c\}^*$. Since $A \cap B = \{c\}$, this says $v \in (A \cap B)^*$. Thus every elt of $A^* \cap B^*$ is also in $(A \cap B)^*$, so $A^* \cap B^* \subset (A \cap B)^*$.
$\supset$: Let v be an arb elt of $(A \cap B)^*$. Since $A \cap B = \{c\}$, this says $v \in \{c\}^*$. Since $c \in A$ and $c \in B$, this tells us that $v \in A^*$ and $v \in B^*$. Thus $v \in A^* \cap B^*$. Thus every elt of $(A \cap B)^*$ is also in $A^* \cap B^*$, so $(A \cap B)^* \subset A^* \cap B^*$. □

Sample Final Exam Solutions

1.

1.	$P \to R$	pr.
2.	$P \to S$	pr.
3.	$\sim P \to G$	pr.
4.	$\sim G$	pr.
5.	P	TI, MT, 3, 4, (DN)
6.	R	det. 1, 5
7.	S	det. 2, 5
8.	$R \wedge S$	rep. 6, 7

2.

1.	C	assum.
2.	$A \to B$	pr.
3.	$\sim A \to \sim C$	pr.
4.	$\sim (C \wedge B)$	pr.
5.	$\sim C \vee \sim B$	EQ, NC, 4
6.	$\sim B$	TI, DS, 1, 5, (DN)
7.	$\sim A$	TI, MT, 2, 6
8.	$\sim C$	det. 3, 7
9.	$C \wedge \sim C$	rep. 1, 8
10.	$C \to (C \wedge \sim C)$	RCP
11.	$\sim C$	TI, IP, 10 (DN)

3.

1.	$\forall x\{\sim G(x) \vee H(x)\}$	pr.
2.	$\forall x G(x) \vee \forall x H(x)$	pr.
3.	$\sim \forall x H(x)$	assum.
4.	$\exists x \sim H(x)$	EQ., NU, 3
5.	$\sim H(a)$	EI, 4
6.	$\sim G(a) \vee H(a)$	UI, 1
7.	$\sim G(a)$	TI, DS, 5, 6
8.	$\exists x \sim G(x)$	EG, 7
9.	$\sim \forall x G(x)$	EQ., NU, 8
10.	$\forall x H(x)$	TI, DS, 2, 9
11.	$H(a)$	UI, 10
12.	$H(a) \wedge \sim H(a)$	rep. 5, 11
13.	$\sim \forall x H(x) \to \{H(a) \wedge \sim H(a)\}$	RCP
14.	$\forall x H(x)$	TI, IP, 13

4.

1.	$\sim \forall x \exists y \{ \mathrm{P}(x,y) \wedge \sim \mathrm{Q}(x,y) \}$	pr.
2.	$\forall x \exists y \{ \mathrm{P}(x,y) \wedge \mathrm{R}(x,y) \}$	pr.
3.	$\exists x \forall y \{ \sim \mathrm{P}(x,y) \vee \mathrm{Q}(x,y) \}$	EQ., NE, NU, NC, (DN), 1
4.	$\forall y \{ \sim \mathrm{P}(a,y) \vee \mathrm{Q}(a,y) \}$	EI, 3
5.	$\exists y \{ \mathrm{P}(a,y) \wedge \mathrm{R}(a,y) \}$	UI, 2
6.	$\mathrm{P}(a,b) \wedge \mathrm{R}(a,b)$	EI, 5
7.	$\sim \mathrm{P}(a,b) \vee \mathrm{Q}(a,b)$	UI, 4
8.	$\mathrm{P}(a,b)$	TI, S, 6
9.	$\mathrm{Q}(a,b)$	TI, DS, 7, 8 (DN)
10.	$\mathrm{R}(a,b)$	TI, S, 6
11.	$\exists y \mathrm{Q}(a,y)$	EG, 9
12.	$\exists x \exists y \mathrm{Q}(x,y)$	EG, 11
13.	$\exists y \mathrm{R}(a,y)$	EG, 10
14.	$\exists x \exists y \mathrm{R}(x,y)$	EG, 13
15.	$\exists x \exists y \mathrm{Q}(x,y) \wedge \exists x \exists y \mathrm{R}(x,y)$	rep., 12, 14

5.

1.	$A \subset \overline{A} \cup \overline{B}$	assum.
2.	$a \in A \cap B$	assum.
3.	$a \in A \wedge a \in B$	EQ., A5, D4, 2
4.	$a \in A$	TI, S, 3
5.	$\forall x (x \in A \rightarrow x \in \overline{A} \cup \overline{B})$	EQ., D1, 1
6.	$a \in A \rightarrow a \in \overline{A} \cup \overline{B}$	UI, 5
7.	$a \in \overline{A} \cup \overline{B}$	det. 4, 6
8.	$a \in \overline{A} \vee a \in \overline{B}$	EQ., A5, D3, 7
9.	$a \notin A \vee a \notin B$	EQ., A5, D2, 8 (twice)
10.	$a \notin B$	TI, DS, 4, 9 (DN)
11.	$a \in B$	TI, S, 3
12.	$a \in B \wedge a \notin B$	rep. 10, 11
13.	$a \in A \cap B \rightarrow (a \in B \wedge a \notin B)$	RCP
14.	$a \notin A \cap B$	TI, IP, 13 (DN)
15.	$\forall x (x \notin A \cap B)$	UG, 14
16.	$A \cap B = \emptyset$	EQ., A4, 15
17.	$A \subset \overline{A} \cup \overline{B} \rightarrow A \cap B = \emptyset$	RCP

6. a) What is A^2? $A^2 = \{a^2, ab, ba, b^2\}$
b) How many elements are in S^2? Ans: 12 ($\lambda, a, b, ab, a^2, a^2b, ba, b^2, bab, aba, ab^2, abab$).
c) Prove by example: $A^* \neq (A^2)^*$ Example: $a \in A^*$, but $a \notin (A^2)^*$ (that is, a is a word over A, but a is not a word over A^2).

7. Let $\mathrm{P}(n)$ be the statement $A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$.
Base - $\mathrm{P}(2) : A \cup (B_1 \cap B_2) = (A \cup B_1) \cap (A \cup B_2)$. True by Thm. 4.24.
Ind. hyp. - Assume $\mathrm{P}(k) : A \cup (B_1 \cap B_2 \cap \cdots \cap B_k) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_k)$, where $k \geq 2$.
Ind. step - Prove $\mathrm{P}(k+1)$: LHS = $A \cup (B_1 \cap B_2 \cap \cdots \cap B_{k+1})$
$= (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_{k+1})$ = RHS.

LHS = $A \cup (B_1 \cap B_2 \cap \cdot\cdot\cdot \cap B_{k+1})$
$= A \cup (B_1 \cap B_2 \cap \cdot\cdot\cdot \cap B_k \cap B_{k+1})$
$= [A \cup (B_1 \cap B_2 \cap \cdot\cdot\cdot \cap B_k)] \cap (A \cup B_{k+1})$ (by Th. 4.24)
$= (A \cup B_1) \cap (A \cup B_2) \cap \cdot\cdot\cdot \cap (A \cup B_k) \cap (A \cup B_{k+1})$ (by ind. hyp.)
$=$ RHS
Thus P$(k+1)$ is true. By PMI, P(n) is true for all $n \geq 2$.
8. A. Let P(n) be the statement $\frac{6^{2n}-1}{7} \in \mathbb{I}$. (Can be translated, $6^{2n} - 1 = 7j$, where $j \in \mathbb{I}$.)
Base - P(1): $\frac{6^2-1}{7} \in \mathbb{I}$. True, since $6^2 - 1 = 35$ and $\frac{35}{7} = 5$.
Ind. hyp. - Assume P(k) : $\frac{6^{2k}-1}{7} \in \mathbb{I}$; that is $6^{2k} - 1 = 7j$, where $j \in \mathbb{I}$. .
Ind. step - Prove P$(k+1)$: $6^{2(k+1)} - 1 \in \mathbb{I}$.
Observe $6^{2(k+1)} - 1 = 6^2 \cdot 6^{2k} - 1 = 36(7j+1) - 1 = 7(36j) + 35 = 7(36j+5)$. Thus
$\frac{6^{2(k+1)}-1}{7} = 36j + 5,$ which $\in \mathbb{I}$. Thus P$(k+1)$ is true.
By PMI, P(n) is true for all $n \geq 1$.
B. Let P(n) be the statement $(n!)^2 < (2n)!$
Base - P(1): $(1!)^2 < (2!)$ $-$ true by observation.
Ind. hyp. - Assume P(k) is true; i.e., $(k!)^2 < (2k)!$, where $k \geq 1$.
Ind. step - Prove P$(k+1)$: LHS = $((k+1)!)^2 < (2(k+1))!$ = RHS
LHS = $((k+1)!)^2 = (k+1)^2(k!)^2$
$< (k+1)^2(2k)!$ (by ind. hyp.)
$< (2k+2)(2k+1)(2k)!$ (since $k+1 < 2k+1 < 2k+2$)
$= (2(k+1))!$ = RHS
Thus P$(k+1)$ is true. By PMI, P(n) is true for all $n \geq 1$.

9. Proof: Let x be an arb. elt. of $(-\infty, 2)$.
Case 1. Let $0 \leq x < 2$. Then $F(x) = \frac{2-x}{x-2} = \frac{-(x-2)}{x-2} = -1$. Also, $G(x) = -1$. Thus $F(x) = G(x)$.
Case 2. Let $x < 0$. Then $F(x) = \frac{2+x}{x-2} = \frac{x-2+4}{x-2} = 1 + \frac{4}{x-2}$. Also, $G(x) = 1 + \frac{4}{x-2}$. Thus $F(x) = G(x)$.
Since $F(x) = G(x)$ in all cases, $F = G$ by Theorem 5.7. ☐

10. Proof: Let x, y, z, w be arb. elts. such that $((x,y),z) \in F$ and $((x,y),w) \in F$. Then $x, y, z, w \in \mathbb{R}$ and $-6 + zx^2 = zy$ and $-6 + wx^2 = wy$. Observe that $x^2 - y \neq 0$. Otherwise, $y = x^2$ and the equation $-6 + zx^2 = zy$ becomes $-6 = 0$, a contradiction.
Thus we may solve for z and w: $z = \frac{6}{x^2-y} = w$. Since x, y, z, w were arb., F is a function. ☐

11. (Hint: To get started on this kind of problem, first try solving for z and notice where a problem might occur. When we solve for z here, we get $z = \frac{2y}{y-x^2}$. There is a division-by-0 problem when $y = x^2$. Assuming $y = x^2$, we can deduce that $x = y = 0$.)
Proof: Note that $((0,0),1)$ and $((0,0),2)$ $\in$ G, but $1 \neq 2$. Thus G is not a function.

12. a) To prove $\text{R}(F) = [0, \infty)$, we use the double subset method.
$\subset$: Let y be an arb. elt. of $\text{R}(F)$. Then $y = |x - 1|$ for some x in $\mathbb{R}$. By the definition of absolute value, $y \geq 0$. Thus $y \in B$. Since y was arb., $\text{R}(F) \subset [0, \infty)$.
$\supset$: Let y be an arb. elt. of B. Define x to be $y + 1$. Then $x \in A$ and $F(x) = |x - 1| = |y + 1 - 1| = y$, since $y \geq 0$. Thus there is an $x \in A$ s.t. $F(x) = y$, so $y \in \text{R}(F)$. Since y was arb., $[0, \infty) \subset \text{R}(F)$.

b) To prove F is not 1-1, we use Template 7. Note that 3 and -1 are two different elements of A and $F(x_1) = F(x_2)$ since $|3 - 1| = |-1 - 1| = 2$. Thus F is not 1-1.

THEOREM SHEET

A1. $\forall x(x \in U)$ (There is a universal set U to which all elements of all sets belong.)

A2. $A = B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B)$

A3. $A = U \leftrightarrow \forall x(x \in A)$

A4. $A = \emptyset \leftrightarrow \forall x(x \notin A)$

A5. If $P(x)$ is an open statement which is meaningful for the elements of U, then $\{x : P(x)\}$ is a set. Also,
$a \in \{x : P(x)\} \leftrightarrow P(a)$ and
$a \notin \{x : P(x)\} \leftrightarrow \sim P(a)$

D1. $A \subset B \leftrightarrow \forall x(x \in A \rightarrow x \in B)$

D2. $\overline{A} = \{x : x \notin A\}$

D3. $A \cup B = \{x : x \in A \vee x \in B\}$

D4. $A \cap B = \{x : x \in A \wedge x \in B\}$

T4.1 $\vdash \forall x(x \notin \emptyset)$

T4.2 $\vdash \emptyset \subset A$

T4.3 $\vdash A \subset U$

T4.4 $\vdash \overline{\emptyset} = U$

T4.5 $\vdash \overline{U} = \emptyset$

T4.6 $\vdash A = B \leftrightarrow (A \subset B \wedge B \subset A)$

T4.7 $\vdash A \subset A \cup B$

T4.8 $\vdash A \cap B \subset A$

T4.9 $\vdash A \cup \emptyset = A$

T4.10 $\vdash \emptyset \cap A = \emptyset$

T4.11 $\vdash U \cup A = U$

T4.12 $\vdash A \cap U = A$

T4.13 $\vdash A \cap \overline{A} = \emptyset$

T4.14 $\vdash A \cup \overline{A} = U$

T4.15 $\vdash \overline{\overline{A}} = A$

T4.16 $\vdash A = B \leftrightarrow \overline{A} = \overline{B}$

T4.17 $\vdash \overline{A \cup B} = \overline{A} \cap \overline{B}$

T4.18 $\vdash \overline{A \cap B} = \overline{A} \cup \overline{B}$

T4.19 $\vdash A \cup B = B \cup A$

T4.20 $\vdash A \cup (B \cup C) = (A \cup B) \cup C$

T4.21 $\vdash A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

T4.22 $\vdash A \cap B = B \cap A$

T4.23 $\vdash A \cap (B \cap C) = (A \cap B) \cap C$

T4.24 $\vdash A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

T4.25 $\vdash A \cap A = A$

T4.26 $\vdash A \cup A = A$

T4.27 $\vdash A \subset A$

T4.28 $\vdash (A \subset B \wedge B \subset C) \rightarrow A \subset C$

T4.29 $\vdash A \subset B \leftrightarrow \overline{B} \subset \overline{A}$

T4.30 $\vdash (A \subset B \wedge C \subset D) \rightarrow A \cup C \subset B \cup D$

T4.31 $\vdash (A \subset B \wedge C \subset D) \rightarrow A \cap C \subset B \cap D$

T4.32 $\vdash A \cap B = A \leftrightarrow A \subset B$

T4.33 $\vdash A \cup B = B \leftrightarrow A \subset B$

D5. $A \underset{\neq}{\subset} B \leftrightarrow (A \subset B \wedge A \neq B)$

T4.34 $\vdash A \underset{\neq}{\subset} B \leftrightarrow \{A \subset B \wedge \exists x(x \in B \wedge x \notin A)\}$

T4.35 $\vdash \sim \{A \underset{\neq}{\subset} A\}$

T4.36 $\vdash A \underset{\neq}{\subset} B \rightarrow \sim \{B \underset{\neq}{\subset} A\}$

T4.37 $\vdash A \underset{\neq}{\subset} B \rightarrow A \subset B$

T4.38 $\vdash (A \underset{\neq}{\subset} B \wedge B \underset{\neq}{\subset} C) \rightarrow A \underset{\neq}{\subset} C$

USEFUL TAUTOLOGIES

1.	$(P \wedge Q) \Leftrightarrow (Q \wedge P)$	Commutative rule for conjunction	CC
2.	$(P \vee Q) \Leftrightarrow (Q \vee P)$	Commutative rule for disjunction	CD
3.	$[P \wedge (Q \wedge R)] \Leftrightarrow [(P \wedge Q) \wedge R]$	Associative rule for conjunction	AC
4.	$[P \vee (Q \vee R)] \Leftrightarrow [(P \vee Q) \vee R]$	Associative rule for disjunction	AD
5.	$[P \wedge (Q \vee R)] \Leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$	Distributive rule for conjunction	DC
6.	$[P \vee (Q \wedge R)] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$	Distributive rule for disjunction	DD
7.	$\sim(\sim P) \Leftrightarrow P$	Double negation	DN
8.	$\sim(P \wedge Q) \Leftrightarrow (\sim P \vee \sim Q)$	Negation of conjunction	NC
9.	$\sim(P \vee Q) \Leftrightarrow (\sim P \wedge \sim Q)$	Negation of disjunction	ND
10.	$\sim(P \Rightarrow Q) \Leftrightarrow (P \wedge \sim Q)$	Negation of implication	NI
11.	$\sim(P \Leftrightarrow Q) \Leftrightarrow (\sim P \Leftrightarrow Q))$	Negation of biconditional	NB
	$\sim(P \Leftrightarrow Q) \Leftrightarrow (P \Leftrightarrow \sim Q)$	Negation of biconditional	NB
12.	$(P \Rightarrow Q) \Leftrightarrow (\sim P \vee Q)$	Implication to disjunction	ID
13.	$(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$	Contrapositive	C
14.	$(P \Leftrightarrow Q) \Leftrightarrow [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$	Biconditional	B
15.	$P \vee \sim P$	Excluded middle	EM
16.	$(P \wedge Q) \Rightarrow P$	Simplification	S
alt.	$(P \wedge Q) \Rightarrow Q$	Simplification	S
17.	$P \Rightarrow (P \vee Q)$	Addition	A
alt.	$Q \Rightarrow (P \vee Q)$	Addition	A
18.	$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$	Modus Ponens	MP
19.	$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$	Hypothetical syllogism	HS
20.	$[(P \vee Q) \wedge \sim Q] \Rightarrow P$	Disjunctive syllogism	DS
alt.	$[(P \vee Q) \wedge \sim P] \Rightarrow Q$	Disjunctive syllogism	DS
21.	$[\sim Q \wedge (P \Rightarrow Q)] \Rightarrow \sim P$	Modus Tollens	MT
22.	$[(P \vee R) \wedge ((P \Rightarrow Q) \wedge (R \Rightarrow S))] \Rightarrow (Q \vee S)$	Constructive dilemma	CDL
23.	$[(\sim Q \vee \sim S) \wedge ((P \Rightarrow Q) \wedge (R \Rightarrow S))] \Rightarrow (\sim P \vee \sim R)$	Destructive dilemma	DDL
24.	$(P \vee P) \Rightarrow P$	Idempotent	IM